An interpolant defined on periodic data: analysis of the error

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Abstract

Given a set of points $x_i, \ i = 0, \ldots, n$ on $[-1, 1]$ and the corresponding values $y_i, \ i = 0, \ldots, n$ of a 2-periodic function $y(x)$, supplied in some way by interpolation or approximation, we describe a simple method that by doubling iteratively this original set, produces in the limit a smooth function. The analysis of the interpolation error is given.

We show that if $y \in \mathcal{C}^4$ then the error in the p-norm, $p = 1, \ p = 2$ and $p = \infty$ depends on the magnitude of the fourth derivative of the function $y(x)$ and on a function $\alpha(x)$ which is even, concave and bounded on $[-1, 1]$.

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1 Introduction

The development of communication systems, for instance computer webs, stimulated the attention on the problem of coding and compressing data. Very frequently in multimedia applications, for example with sound data, as coding functional one chooses the values of a function on a given set of points or the values of the functions on a finer data set obtained from the original by splitting each interval or doubling the number of the data points. To regenerate the underlying function in the literature there exist many methods based on wavelets, interpolation by trigonometric or algebraic polynomials or by using splines (cf. [5, 2]).

In the present paper we address the problem of interpolating a set of specific values supplied in some ways and by an interative-like interpolation process we build the underlying interpolant. Such problems often arise in mechanical engineering, for the necessity of numerical control metalcutting, in problems related to computer modelling and many other problems of processing curves and surfaces (cfr. [5]).

Let us start by formulating our problem. Let $y_i = y_{i,0},\ i = 0, \ldots, n$ be the values of a 1-periodic function $f(x) : [a, b] \to \mathbb{R}$ at the equispaced points $x_{i,0} = ih,\ h = \frac{b-a}{n}$. In what follow, we assume for the sake of simplicity that $[a, b] = [-1, 1]$, and that the points are supplied in some way: for instance they can be the data of some interpolating or approximating process (Lagrange or spline interpolation, smoothing spline, etc...) which we consider as *initial* data.

The method builds the interpolant simply by doubling iteratively the original set of points. The analysis of the interpolation error is discussed with the aid of a 2-periodic function $\alpha(x)$ which is shown to be bounded and concave on $[-1, 1]$.

2 The method

The idea of the method is as follows. Given the data $y_{i,0},\ i = 0, ..., n$ the new supplied values are obtained by inserting at each middle point a new value, generated by an interpolation process, getting $y_{i,1},\ i = 0, \ldots, 2n$. Now we proceed similarly getting $y_{i,2},\ i = 0, \ldots, 4n$ and so forth.

2.1 The interpolation process

Starting from the initial set $(x_{i,0}, y_{i,0}),\ i = 0, ..., n$, through the points $(x_{\nu,0}, y_\nu)$ $\nu = i-1, i, i+1, i+2, \quad 1 \le i \le n-2$ we construct the interpolating cubic polynomial.

Now, let $\hat{y}_{i+\frac{1}{2},0}$ be the value of $y(x)$ at the point $x_{i+\frac{1}{2},0}$, that is:

$$\begin{aligned}
\hat{y}_{i+\frac{1}{2},0} &= \frac{1}{16}\left(-y_{i-1,0} + 9y_{i,0} + 9y_{i+1,0} - y_{i+2,0}\right) \\
&= \frac{y_{i+1,0} + y_{i,0}}{2} - \frac{1}{8}\Delta^2\left(\frac{y_{i+1,0} + y_{i,0}}{2}\right), \qquad (1)
\end{aligned}$$

where $\Delta^2 z_i = z_{i+1} - 2z_i + z_{i-1}$ stands for the second central difference at the point z_i.

Letting

$$y_{2i,1} = y_{i,0} \ , \qquad i = 0, \dots, n, \tag{2}$$

$$y_{2i+1,1} = \hat{y}_{i+\frac{1}{2},0} \ , \qquad i = 0, \dots, n-1. \tag{3}$$

then we have provided a way to compute the function values $y_{i,1}, \ i = 0, \dots, \ 2n$.

Remarks. Looking to the definition of the scheme, the points $y_{1,1}$ and $y_{2n-1,1}$ can not be determined with the scheme since they require the knowledge of two points outside $[a, b]$, let us say $y_{-1,0}$ and $y_{n+1,0}$. To determine them we can follow two directions:

1. define them as the values of the interpolating cubic spline passing through the points $(x_{0,0}, y_{0,0}), \ \ (x_{1,0}, y_{1,0})$ and $(x_{n-1,0}, y_{n-1,0}), \ \ (x_{n,0}, y_{n,0})$, respectively;

2. by using Newton's interpolating formula we have

$$\begin{aligned} y_{-1,0} &= 5y_{0,0} - 10y_{1,0} + 10y_{2,0} - 5y_{3,0} + y_{4,0} \ , \\ y_{n+1,0} &= 5y_{n,0} - 10y_{n-1,0} + 10y_{n-2,0} - 5y_{n-3,0} + y_{n-4,0} \ . \end{aligned}$$

As well-known, Newton's formula gives a global polynomial interpolant which can oscillate close to the end points of [a,b]. Therefore, the use of cubic splines, which are piecewise polynomials, can sensibly improve the fitting, as shown in the Figure 1.

Then, by using (1) and the previous remarks, we are able to compute

$$\hat{y}_{i+\frac{1}{2},1} = \frac{y_{i+1,1} + y_{i,1}}{2} - \frac{1}{8}\Delta^2 \left(\frac{y_{i+1,1} + y_{i,1}}{2} \right) \tag{4}$$

that is the values $y_{i,2}, \ i = 0, \dots, 2^2 \, n$.

Finally the recurrence required is:

$$\hat{y}_{i+\frac{1}{2},k-1} = \frac{y_{i+1,k-1} + y_{i,k-1}}{2} - \frac{1}{8}\Delta^2 \left(\frac{y_{i+1,k-1} + y_{i,k-1}}{2} \right), \tag{5}$$

and

$$y_{2i,k} = y_{i,k-1} \ , \qquad i = 0, \dots, 2^{k-1} n,$$

$$y_{2i+1,k} = \hat{y}_{i+\frac{1}{2},k-1} \ , \qquad i = 0, \dots, 2^{k-1} n - 1,$$

where k is any natural number.

For an example of how the scheme is working see below, Section 5 where a comparison with the well-known *Shepard's method for data-fitting* is provided.

We notice that this interpolation scheme is similar to the so-called 9/16-subdvision scheme studied by Dubuc [3] and generalized to the bidimensional case by S. De Marchi [1].

3 The function $\alpha(x)$ and its properties

To study the error of the method we first introduce the following function $\alpha(x)$.
We assume $\alpha(-1) = \alpha(1) = 0$ and

$$\alpha(0) = \left(\frac{1}{16}\right)^0 + \frac{1}{2}\left(\alpha(-1) + \alpha(1)\right) = 1. \tag{6}$$

Assuming that $\alpha\left(\nu/2^{k-1}\right)$ and $\alpha\left((\nu+1)/2^{k-1}\right)$ are known, then we may define $\alpha(x)$ as a recurrence:

$$\alpha\left(\frac{2\nu+1}{2^k}\right) = \left(\frac{1}{16}\right)^k + \frac{1}{2}\left(\alpha\left(\frac{\nu}{2^{k-1}}\right) + \alpha\left(\frac{\nu+1}{2^{k-1}}\right)\right), \tag{7}$$

where $\nu = -2^{k-1}, \ldots, 2^{k-1} - 1$, $k \in \mathbb{N}$, $k \geq 0$.
For each fixed k we shall define a step-function $\hat{\alpha}_k(x)$, $x \in [-1, 1]$ by the relation

$$\hat{\alpha}_k(x) = \alpha\left(\frac{\nu}{2^k}\right) \qquad \left(x \in \left(\frac{\nu}{2^k}, \frac{\nu+1}{2^k}\right), \nu = -2^k, \ldots, 2^k - 1\right)$$

and at each point of discontinuity we set it by the following:

$$\hat{\alpha}_k\left(\frac{\nu}{2^k}\right) = \frac{1}{2}\left(\hat{\alpha}_k\left(\frac{\nu}{2^k} + 0\right) + \hat{\alpha}_k\left(\frac{\nu}{2^k} - 0\right)\right).$$

It is possible to show (see later the Corollary 1 and (12)) that there exists a unique limit as $k \to \infty$ for the function $\hat{\alpha}_k(x)$ and this limit holds uniformly. Let $\alpha(x)$ be this limit.

Lemma 1 *For any fixed k the following inequality holds:*

$$\Delta^2\alpha\left(\frac{\nu}{2^k}\right) < 0, \quad \nu = -2^k + 1, \ldots, 2^k - 1.$$

Proof. From equation (7)

$$-\left(\frac{1}{16}\right)^k = \frac{1}{2}\left(\alpha\left(\frac{\nu}{2^{k-1}}\right) + \alpha\left(\frac{\nu+1}{2^{k-1}}\right)\right) - \alpha\left(\frac{2\nu+1}{2^k}\right),$$

For odd ν the Lemma is true, since

$$-2\left(\frac{1}{16}\right)^k = \Delta^2\alpha\left(\frac{2\nu+1}{2^k}\right). \tag{8}$$

For even ν we proceed by mathematical induction.
As follows from (8) we have $\Delta^2\alpha\left(\frac{1}{2}\right) \leq -2$.

Now, for all ν, $\nu = -2^{k-2}, \dots, 2^{k-2} - 1$ the following inequality holds:

$$\Delta^2 \alpha \left(\frac{2(2\nu + 1)}{2^{k-1}} \right) \leq -2 \left(\frac{1}{16} \right)^{k-1}, \tag{9}$$

To conclude we should prove that for all ν, $\nu = -2^{k-1}, \dots, 2^{k-1} - 1$

$$\Delta^2 \alpha \left(\frac{4(2\nu + 1)}{2^k} \right) \leq -2 \left(\frac{1}{16} \right)^k .$$

Using (7), we obtain

$$\Delta^2 \alpha \left(\frac{4(2\nu + 1)}{2^k} \right) = 2 \left(\frac{1}{16} \right)^k +$$

$$+\frac{1}{2} \left(\alpha \left(\frac{2(2\nu + 1) - 1}{2^{k-1}} \right) + \alpha \left(\frac{2(2\nu + 1) + 1}{2^{k-1}} \right) \right) - \alpha \left(\frac{2\nu + 1}{2^{k-2}} \right) =$$

$$= 2 \left(\frac{1}{16} \right)^k + \frac{1}{2} \left(\alpha \left(\frac{4\nu + 1}{2^{k-1}} \right) - 2\alpha \left(\frac{4\nu + 2}{2^{k-1}} \right) + \alpha \left(\frac{4\nu + 3}{2^{k-1}} \right) \right) .$$

Therefore from this results and from (9) we finally have

$$\Delta^2 \alpha \left(\frac{4(2\nu + 1)}{2^k} \right) \leq 2 \left(\frac{1}{16} \right)^k - \left(\frac{1}{16} \right)^{k-1} \leq -2 \left(\frac{1}{16} \right)^k .$$

This concludes our proof. $\square$

Corollary 1 *From Lemma 1 and the fact that*

$$\alpha \left(\frac{\nu}{2^k} \right) = \alpha \left(-\frac{\nu}{2^k} \right)$$

the function $\alpha(x)$ is even, concave and the following relations hold:

$$\max_{\nu = -2^k, \dots, 2^k} \alpha \left(\frac{\nu}{2^k} \right) = \max_{x \in [-1, 1]} \alpha(x) = \alpha(0) = 1.$$

$\diamond\diamond$

The following Lemma specializes the Lemma 1.

Lemma 2 *Let $\mu \leq k - 1$, then*

$$\Delta^2 \alpha \left(\frac{2^\mu (2\nu + 1)}{2^k} \right) = -\frac{1}{7} \left(\frac{1}{2} \right)^{4k-1} \left(6 \cdot 2^{3\mu} + 1 \right) \tag{10}$$

Proof. For $\mu=0$ the statement follows from (8).
On the other side, it is clear that

$$\Delta^2\alpha\left(\frac{2^\mu(2\nu+1)}{2^k}\right)=\alpha\left(\frac{2^\mu(2\nu+1)-1}{2^k}\right)-2\alpha\left(\frac{2^\mu(2\nu+1)}{2^k}\right)+\alpha\left(\frac{2^\mu(2\nu+1)+1}{2^k}\right).$$

By using (7), we also get

$$\begin{aligned}\Delta^2\alpha\left(\frac{2^\mu(2\nu+1)}{2^k}\right) &= 2\left(\frac{1}{16}\right)^k+\frac{1}{2}\left[\alpha\left(\frac{2^{\mu-1}(2\nu+1)-1}{2^{k-1}}\right)+\alpha\left(\frac{2^{\mu-1}(2\nu+1)+1}{2^{k-1}}\right)\right]- \\ &- \alpha\left(\frac{2\nu+1}{2^{k-\mu}}\right).\end{aligned} \tag{11}$$

Repeating this process μ times, we have

$$\Delta^2\alpha\left(\frac{2^\mu(2\nu+1)}{2^k}\right)=2\left(\frac{1}{16}\right)^k+\left(\frac{1}{16}\right)^{k-1}+\ldots+\frac{1}{2^{\mu-2}}\left(\frac{1}{16}\right)^{k-\mu-1}+$$

$$+\frac{1}{2^\mu}\left(\alpha\left(\frac{(2\nu+1)-1}{2^{k-\mu}}\right)+\alpha\left(\frac{(2\nu+1)+1}{2^{k-\mu}}\right)\right)-\frac{1}{2^{\mu-1}}\alpha\left(\frac{2\nu+1}{2^{k-\mu}}\right)=$$

$$=\left(\frac{1}{2}\right)^{4k+1}\sum_{j=0}^{\mu-1}8^j+\frac{1}{2^\mu}\Delta^2\alpha\left(\frac{2\nu+1}{2^{k-\mu}}\right).$$

Then from (8) it follows

$$\Delta^2\alpha\left(\frac{2^\mu(2\nu+1)}{2^k}\right)=\left(\frac{1}{2}\right)^{4k+1}\sum_{j=0}^{\mu-1}8^j-2\frac{1}{2^\mu}\left(\frac{1}{16}\right)^{k-\mu}.$$

Noticing, that

$$\sum_{j=0}^{\mu}8^j=\frac{8^\mu-8}{8-1}=\frac{8^\mu-8}{7},$$

we obtain

$$\Delta^2\alpha\left(\frac{2^\mu(2\nu+1)}{2^k}\right)=\left(\frac{1}{2}\right)^{4k-1}\frac{8^{\mu+1}-8}{7}-\left(\frac{1}{2}\right)^{4k-3\mu+1}=$$

$$=\left(\frac{1}{2}\right)^{4k-1}\left(\frac{8^\mu-1}{7}-8^\mu\right)=\frac{1}{7}\left(\frac{1}{2}\right)^{4k-1}\left(-6\cdot 8^\mu-1\right).$$

This concludes the proof. □

Corollary 2

$$\max_{\nu=-2^k+1,\ldots,2^k-1}\left|\Delta^2\alpha\left(\frac{\nu}{2^k}\right)\right|=\frac{1}{7}\left(\frac{1}{2}\right)^{4k-1}\left(6\cdot 8^{k-1}+1\right).$$

and

$$\lim_{k\to\infty}\Delta^2\alpha\left(\frac{\nu}{2^k}\right)=0. \tag{12}$$

$\diamond\diamond$

Let $\tilde{\alpha}_k(x)$ be the *linear interpolant* at the point set

$$\left(\frac{\nu}{2^k}, \alpha\left(\frac{\nu}{2^k}\right)\right) \qquad \nu = -2^k, \ldots, 2^k.$$

Then

$$\alpha(x) = \lim_{k\to\infty} \tilde{\alpha}_k(x).$$

Let

$$\|f\|_{p\,,[a,b]} = \begin{cases} \left(\frac{1}{b-a}\int_a^b |f(t)|^p dt\right)^{1/p} & p \in [1,\infty) \\ \\ \max_{t\in[a,b]} |f(t)| & p = \infty, \end{cases}$$

When $[a,b] = [-1,1]$, we shall simply write $\|f\|_p = \|f\|_{p\,,[-1,1]}$.

Lemma 3 *Let $k = 1, 2, \ldots$ then*

$$\|\alpha_k\|_\infty = 1, \tag{13}$$

$$\|\alpha_k\|_1 = \frac{8}{15}\left(1 - \frac{1}{16^{k+1}}\right), \tag{14}$$

$$\|\alpha_k\|_2^2 = \frac{64}{225}\left(1 - \frac{1}{16^{k+1}}\right)^2 + \frac{64}{765}\left(1 - \frac{1}{16^{2k+2}}\right). \tag{15}$$

Therefore, $\|\alpha\|_1 = 0.533$, $\|\alpha\|_2 = \sqrt{\frac{1408}{3825}} \approx 0.607$, $\|\alpha\|_\infty = 1$ *and*

$$\|\alpha\|_1 < \|\alpha\|_2 < \|\alpha\|_\infty.$$

Proof. From the Corollary 1 we have at once the equality (13).

Let us consider the 2-periodic function

$$\tilde{\alpha}_0(x) = \begin{cases} x+1, & x \in [-1,0] \\ 1-x, & x \in (0,1] \end{cases}$$

and

$$\tilde{\alpha}_k(x) = \tilde{\alpha}_0(x) + \frac{1}{16}\tilde{\alpha}_0(2x) + \frac{1}{16^2}\tilde{\alpha}_0(4x) + \ldots + \frac{1}{16^k}\tilde{\alpha}_0(2^k x) = \sum_{\nu=0}^{k} \frac{1}{16^\nu}\tilde{\alpha}_0(2^\nu x). \tag{16}$$

Thus

$$\|\alpha_k\|_1 = \frac{1}{2}\int_{-1}^{1}\sum_{\nu=0}^{k}\frac{1}{16^\nu}\tilde{\alpha}_0(2^\nu x)\,dx = \frac{1}{2}\int_{-1}^{1}\sum_{\nu=0}^{k}\int_{-1}^{1}\frac{1}{16^\nu}\tilde{\alpha}_0(2^\nu x)\,dx =$$

$$= \frac{1}{2}\int_{-1}^{1}\sum_{\nu=0}^{k}\int_{-1}^{1}\frac{1}{16^\nu}\tilde{\alpha}_0(x)\,dx = \frac{8}{15}\left(1-\left(\frac{1}{16}\right)^{k+1}\right).$$

At once we have

$$\|\alpha\|_1 = \frac{8}{15}.$$

We remind that 2-periodic even continuous function $f(x)$ can to be expanded in Fourier series, that is

$$f(x) = a_0 + \sum_{\nu=1}^{\infty} a_\nu \cos(\nu\pi x)$$

and the Parseval's identity

$$\int_{-1}^{1} f^2(x)dx = 2\,a_0^2 + \sum_{\nu=1}^{\infty} a_\nu^2. \tag{17}$$

As

$$\tilde{\alpha}_0(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{i=1}^{\infty}\frac{\cos((2i+1)\pi x)}{(2i+1)^2},$$

then from (16) we have

$$\tilde{\alpha}_k(x) = \sum_{\nu=0}^{k}\frac{1}{16^\nu}\left(\frac{1}{2}+\frac{4}{\pi^2}\sum_{i=1}^{\infty}\frac{\cos((2i+1)2^\nu\pi x)}{(2i+1)^2}\right) =$$

$$= \frac{1}{2}\sum_{\nu=0}^{k}\frac{1}{16^\nu} + \frac{4}{\pi^2}\sum_{\nu=0}^{k}\frac{1}{16^\nu}\sum_{i=1}^{\infty}\frac{\cos((2i+1)2^\nu\pi x)}{(2i+1)^2} =$$

$$= \frac{8}{15}\left(1-\frac{1}{16^{k+1}}\right) + \frac{4}{\pi^2}\sum_{\nu=0}^{k}\sum_{i=1}^{\infty}\frac{\cos((2i+1)2^\nu\pi x)}{16^\nu(2i+1)^2}. \tag{18}$$

Let us observe, that for $i, m \in \mathbb{N}$

$$(2i+1)2^\nu = (2j+1)2^\mu \iff i=j,\ \nu=\mu.$$

If $\nu > \mu$, then

$$(2i+1)2^\nu = (2j+1)2^\mu \iff (2i+1)2^{\nu-\mu} = (2j+1).$$

Therefore from this fact, since in the left side of (18) each item meets only once, then by using the Parseval's identity

$$\int_{-1}^{1}\tilde{\alpha}_k^2(x)\,dx = 2\left(\frac{8}{15}\left(1-\frac{1}{16^{k+1}}\right)\right)^2 + \frac{16}{\pi^4}\sum_{\nu=0}^{k}\sum_{i=0}^{\infty}\frac{1}{16^{2\nu}}\frac{1}{(2i+1)^4}.$$

Noting, that

$$\sum_{i=0}^{\infty}\frac{1}{(2i+1)^4} = \frac{\pi^4}{96},$$

we get (15). $\square$

4 The analysis of the error of the interpolating method

Given the data $x_i = x_{i,0} = ih, \ \ i = 0, 1, \ldots, n$ we set

$$x_{i+\frac{1}{2}} = \left(i + \frac{1}{2}\right) h$$

and

$$x_{i,k} = i\frac{h}{2^k}\ , \qquad i = 0, \ldots, 2^k n.$$

Moreover, let $\tilde{y}_k(x)$ be the piecewise linear function, interpolating the values $y_{i,k}$ at the points $x_{i,k}$, $i = 0, \ldots, 2^k n$.

Our main result is the following Theorem:

Theorem 1 *Let $y \in \mathcal{C}^4$ and let k be any fixed natural number, then*

$$\|y - \tilde{y}_k\|_p \leq \frac{3}{128} h^4 \|\alpha\|_p \|y^{(4)}\|_p + \mathcal{O}(h^5) \tag{19}$$

where

$$\|\alpha\|_\infty = 1,$$

$$\|\alpha\|_1 = \frac{8}{15},$$

$$\|\alpha\|_2 = \sqrt{\frac{1408}{3825}}.$$

Before proving the Theorem we wish to prove a preliminary statement.

Lemma 4 *Let $y \in \mathcal{C}^4$, $n, k \in \mathbb{N}$ and $\nu = 2^k i + \mu$, $i = 0, \ldots, n-1$, $\ \mu = 0, 1, \ldots, 2^k$. Then uniformly on ν*

$$y_{\nu,k} = y(x_{\nu,k}) - \gamma h^4 \alpha\left(\frac{j}{2^{k-1}}\right) y^{(4)}(x_{\nu,k}) + \mathcal{O}(h^5) \quad \nu = 0, 1, \ldots, 2^k n, \tag{20}$$

where

$$\gamma = \frac{3}{128} \qquad \textit{and} \qquad j = \mu - 2^{k-1}.$$

Proof. By induction on k.
Let us consider the Taylor expansion for the function $y(x)$ around the point $x_{i+\frac{1}{2}}$, we have

$$\frac{y_{i+1,0} + y_{i,0}}{2} = y(x_{i+\frac{1}{2}}) + \frac{1}{2}\left(\frac{h}{2}\right)^2 y''(x_{i+\frac{1}{2}}) + \frac{1}{4!}\left(\frac{h}{2}\right)^4 y^{(4)}(x_{i+\frac{1}{2}}) + \mathcal{O}(h^5).$$

Then, substituting in (1), we have

$$\hat{y}_{i+\frac{1}{2},0} = y(x_{i+\frac{1}{2}}) + \frac{1}{2}\left(\frac{h}{2}\right)^2 y''(x_{i+\frac{1}{2}}) + \frac{1}{4!}\left(\frac{h}{2}\right)^4 y^{(4)}(x_{i+\frac{1}{2}}) + \mathcal{O}(h^5) -$$

$$-\frac{1}{8}\Delta^2\left(y(x_{i+\frac{1}{2}})+\frac{1}{2}\left(\frac{h}{2}\right)^2 y''(x_{i+\frac{1}{2}})+\frac{1}{4!}\left(\frac{h}{2}\right)^4 y^{(4)}(x_{i+\frac{1}{2}})+\mathcal{O}(h^5)\right)=$$

$$=y(x_{i+\frac{1}{2}})-\frac{3}{128}h^4 y^{(4)}(x_{i+\frac{1}{2}})+\mathcal{O}(h^5) \tag{21}$$

Thus, if the values $y_{i,0}$ and $y_{i+1,0}$ are known, the values $\hat{y}_{i+\frac{1}{2}}$ are determined through the values of the function $y(x_{i+\frac{1}{2}})$ as follows

$$y_{2i+1,1}=y(x_{i+\frac{1}{2}})-\gamma h^4\alpha(0)y^{(4)}(x_{i+\frac{1}{2}})+\mathcal{O}(h^5). \tag{22}$$

Continuing this process

$$\hat{y}_{2i+\frac{1}{2},1}=\frac{y_{2i+1,1}+y_{2i,1}}{2}-\frac{1}{8}\Delta^2\left(\frac{y_{2i+1,1}+y_{2i,1}}{2}\right)$$

Noticing that $y_{2i,1}=y_{i,0}$, we obtain

$$\hat{y}_{2i+\frac{1}{2},1}=\frac{y(x_{i+\frac{1}{2}})-\gamma h^4\alpha(0)y^{(4)}(x_{i+\frac{1}{2}})+y_{i,0}}{2}-$$

$$-\frac{1}{8}\Delta^2\left(\frac{y(x_{i+\frac{1}{2}})+y_{i,0}}{2}\right)+\mathcal{O}(h^5)=$$

$$=\frac{y(x_{i+\frac{1}{2}})+y_{i,0}}{2}-\frac{\gamma}{2}h^4\alpha(0)y^{(4)}(x_{i+\frac{1}{2}})-$$

$$-\frac{1}{8}\Delta^2\left(\frac{y_{i+\frac{1}{2},0}+y_{i,0}}{2}\right)+\mathcal{O}(h^5)=$$

$$=\frac{y(x_{i+\frac{1}{2}})+y_i}{2}-\frac{1}{8}\Delta^2\left(\frac{y_{i+\frac{1}{2},0}+y_{i,0}}{2}\right)-\frac{\gamma}{2}h^4\alpha(0)y^{(4)}(x_{i+\frac{1}{2}})+\mathcal{O}(h^5).$$

Since $y(x_{i+\frac{1}{2}})$ comes from x_i but using $h/2$ instead of h, then

$$\hat{y}_{2i+\frac{1}{2},1}=y_{i+\frac{1}{4},0}-\frac{3}{128}\left(\frac{h}{2}\right)^4 y^{(4)}_{i+\frac{1}{4}}-\frac{\gamma}{2}h^4\alpha(0)y^{(4)}_{i+\frac{1}{4}}+\mathcal{O}(h^5)=$$

$$=y_{i+\frac{1}{4},0}-\gamma h^4\left(\left(\frac{1}{2}\right)^4+\frac{1}{2}\alpha(0)\right)y^{(4)}_{i+\frac{1}{4}}+\mathcal{O}(h^5).$$

Noticing, that

$$\alpha\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^4+\frac{1}{2}(\alpha(0)+\alpha(1)),$$

we have

$$\hat{y}_{2i+\frac{1}{2},1}=y_{i+\frac{1}{4},0}-\gamma h^4\alpha\left(\frac{1}{2}\right)y^{(4)}_{i+\frac{1}{4}}+\mathcal{O}(h^5).$$

To conclude we shall make one more step. The error at the point $(i+\frac{3}{8})h$ is

$$\hat{y}_{i+\frac{3}{8}} = \frac{\hat{y}_{i+\frac{1}{4}} + \hat{y}_{i+\frac{1}{2}}}{2} - \frac{1}{8}\Delta^2 \left(\frac{\hat{y}_{i+\frac{1}{4}} + \hat{y}_{i+\frac{1}{2}}}{2} \right).$$

Therefore, we get

$$\hat{y}_{i+\frac{3}{8}} = \frac{y(x_{i+\frac{1}{2}}) - \gamma h^4 \alpha(0) y^{(4)}(x_{i+\frac{1}{2}}) + y_{i+\frac{1}{4}} - \gamma h^4 \alpha\left(\frac{1}{2}\right) y^{(4)}_{i+\frac{1}{4}}}{2} -$$

$$-\frac{1}{8}\Delta^2 \left(\frac{y(x_{i+\frac{1}{2}}) + y_{i+\frac{1}{4}}}{2} \right) + \mathcal{O}(h^5) =$$

$$= \frac{y_{i+\frac{1}{2},0} + y_{i+\frac{1}{4},0}}{2} - \frac{1}{8}\Delta^2 \left(\frac{y_{i+\frac{1}{2},0} + y_{i+\frac{1}{4},0}}{2} \right) -$$

$$-h^4 \frac{\gamma}{2} \left(\alpha\left(\frac{1}{2}\right) y^{(4)}_{i+\frac{1}{4}} + \alpha(0) y^{(4)}(x_{i+\frac{1}{2}}) \right) + \mathcal{O}(h^5).$$

Since the values $y_{i+\frac{1}{2},0}$ and $y_{i,0}$ are points of the function, then as follows from (21) (using $\frac{h}{4}$ instead of h)

$$\hat{y}_{i+\frac{3}{8}} = y_{i+\frac{3}{8}} - \frac{3}{128}\left(\frac{h}{4}\right)^4 y^{(4)}_{i+\frac{3}{8}} - \gamma h^4 \frac{1}{2} \left(\alpha\left(\frac{1}{2}\right) y^{(4)}_{i+\frac{1}{4}} + \alpha(0) y^{(4)}(x_{i+\frac{1}{2}}) \right) + \mathcal{O}(h^5) =$$

$$= y_{i+\frac{3}{8}} - \gamma h^4 \left(\left(\frac{1}{4}\right)^4 + \frac{1}{2} \left(\alpha\left(\frac{1}{2}\right) + \alpha(0) \right) y^{(4)}_{i+\frac{3}{8}} \right) + \mathcal{O}(h^5) =$$

$$= y_{i+\frac{3}{8}} - \gamma h^4 \alpha\left(\frac{1}{4}\right) y^{(4)}_{i+\frac{3}{8}} + \mathcal{O}(h^5).$$

Hence

$$y_{\nu,k-1} = y(x_{\nu,k-1}) - \gamma\alpha\left(\frac{j}{2^{k-2}}\right) y^{(4)}(x_{\nu,k-1}) + \mathcal{O}(h^5).$$

Then from (4) we have

$$y_{2\nu+1,k} = \frac{y_{\nu+1,k-1} + y_{\nu,k-1}}{2} - \frac{1}{8}\Delta^2 \left(\frac{y_{\nu+1,k-1} + y_{\nu,k-1}}{2} \right) =$$

$$= \frac{1}{2} \left(y(x_{\nu+1,k-1}) - \gamma\alpha\left(\frac{j+1}{2^{k-2}}\right) y^{(4)}(x_{\nu+1,k-1}) + y(x_{\nu,k-1}) - \gamma\alpha\left(\frac{j}{2^{k-2}}\right) y^{(4)}(x_{\nu,k-1}) \right) -$$

$$-\frac{1}{8}\Delta^2 \left(\frac{y_{\nu+1,k-1} + y_{\nu,k-1}}{2} \right) + \mathcal{O}(h^5) = \frac{1}{2} \left(y(x_{\nu,k-1}) + y(x_{\nu+1,k-1}) \right) -$$

$$-\frac{\gamma}{2} \left(\alpha\left(\frac{j}{2^{k-2}}\right) y^{(4)}(x_{2\nu+1,k}) + \alpha\left(\frac{j+1}{2^{k-2}}\right) y^{(4)}(x_{2\nu+1,k}) \right) -$$

$$-\frac{1}{8}\Delta^2\left(\frac{y(x_{\nu,k-1})+y(x_{\nu+1,k-1})}{2}\right)+\mathcal{O}(h^5).$$

As follows from (1) and (21),

$$\frac{1}{2}\left(y(x_{\nu,k-1})+y(x_{\nu+1,k-1})\right)-\frac{1}{8}\Delta^2\left(\frac{y(x_{\nu,k-1})+y(x_{\nu+1,k-1})}{2}\right)=$$

$$=y(x_{2\nu+1,k})-\frac{3}{128}\left(\frac{h}{2^k}\right)^4 y^{(4)}(x_{2\nu+1,k})+\mathcal{O}(h^5).$$

Thus we have

$$y_{2\nu+1,k}=y(x_{2\nu+1,k})-h^4\gamma\left(\frac{1}{2}\right)^{4k}y^{(4)}(x_{2\nu+1,k})-$$

$$-\frac{\gamma}{2}h^4\left(\alpha\left(\frac{j}{2^{k-2}}\right)+\alpha\left(\frac{j+1}{2^{k-2}}\right)\right)y^{(4)}(x_{2\nu+1,k})+\mathcal{O}(h^5)=$$

$$=y(x_{2\nu+1,k})-\gamma h^4\alpha\left(\frac{2j+1}{2^{k-1}}\right)y^{(4)}(x_{2\nu+1,k})+\mathcal{O}(h^5).$$

From here and from (7) we obtain (20). □

The proof of the Theorem 1 is now an application of the the Lemma 3 and 4. □

5 Example

We conclude our investigations with a simple but instructive example. Let us consider the function

$$y(x)=\frac{10}{1+100\,x^2}\quad x\in[-1,1]\ ,$$

which is a good test function since it has a rapidly varying gradient.

Starting from an initial set of data, we firstly built the interpolant using our interpolation scheme. A picture of the results obtained just after 2 steps are displayed in Fig. 1.

Then we made a comparison of our scheme with the well-known Shepard's method for data-fitting, as described in [4]. The method is simply an interpolating moving least-squares method, with weight functions

$$w_i(x)=\frac{1}{(x-x_i)^s},\quad s>0$$

where the x_i are the interpolation points.

From Figures 2-4 it is clear that our method is much more precise than the Shepard's one. For instance, the Shepard's method fails to fit the data when $s=1$. When $s=2$ the flat-spot phenomenon starts and this becomes more evident for $s>2$. A first remark is that while our method produces a cubic piecewise interpolant which well fit the data just after a few steps, the Shepard's method does not (as discussed in [4]).

In order to analyse the error we need some settings. Letting $\varepsilon_k(y;x) = y(x) - \tilde{y}_k(x)$ and

$$\varepsilon_{k,h,p}(y) = \left| \|\varepsilon_k(y)\|_p - \frac{3}{128} h^4 \|\alpha\|_p \|y^{(4)}\|_p \right| .$$

Then, using the sup-norm and after $k = 10$ iterated we have:

$$\left\| \varepsilon_{10} \left(\frac{10}{1 + 100(\cdot)^2} \right) \right\|_\infty \approx 1.7326 \cdot 10^{-2} .$$

We observe that this is nothing else that the error due to the approximation of a piecewise linear interpolation

$$\|\varepsilon_k(y)\ |_p = \frac{1}{8} \left(\frac{h}{2^k} \right)^2 \ |y^{(2)}\|_p + \mathcal{O} \left(\frac{h}{2^k} \right)^3 .$$

Thus at the values k such that

$$\|\varepsilon_k(y)\|_p = \frac{1}{8} \left(\frac{h}{2^k} \right)^2 \|y^{(2)}\|_p + O \left(\frac{h}{2^k} \right)^3 .$$

for which

$$\left(\frac{h}{2^k} \right)^2 \gg h^4,$$

hence

$$k \gg \log_2 h$$

which implies that suitable values for k are so that $k \gg 8$.

Finally

$$\varepsilon_{10,\frac{1}{5},\infty} \left(\frac{10}{1 + 100(\cdot)^2} \right) \approx 5.87 \cdot 10^{-3}.$$

Remarks.

1. The constant can essentially be reduced. This example is a test example of approximation of bad function with a quickly varying gradient.

2. Constant it is possible and it is necessary to reduce but this step will require more investigations.

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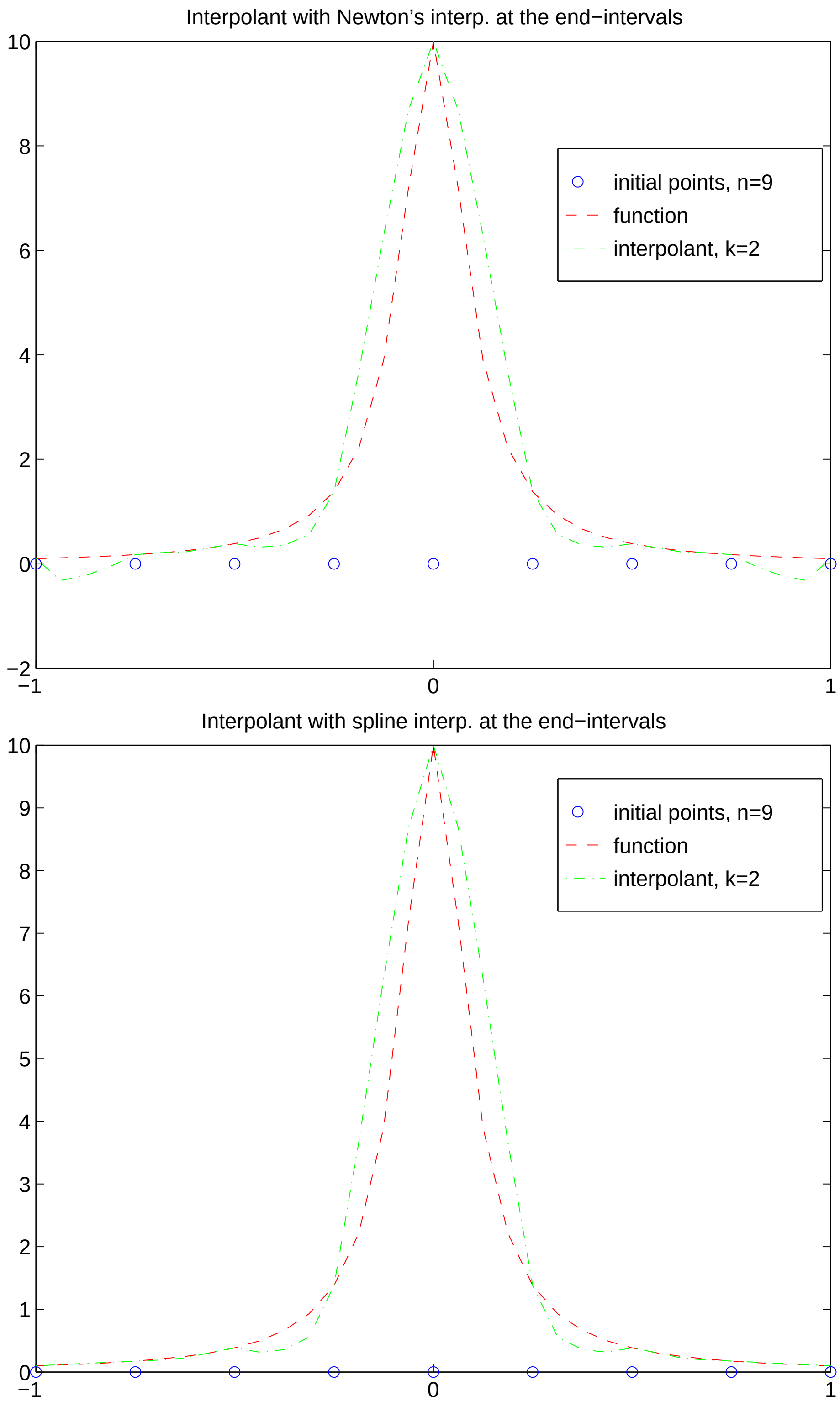

Figure 1: The function $y(x) = 1/(1+100\,x^2)$ on [-1,1] (solid line) and the interpolant after $k = 2$ steps (dashed line) as obtained with our scheme. Above by using Newton's interpolation to compute the first and last new inserted point, while below by using cubic splines interpolation.

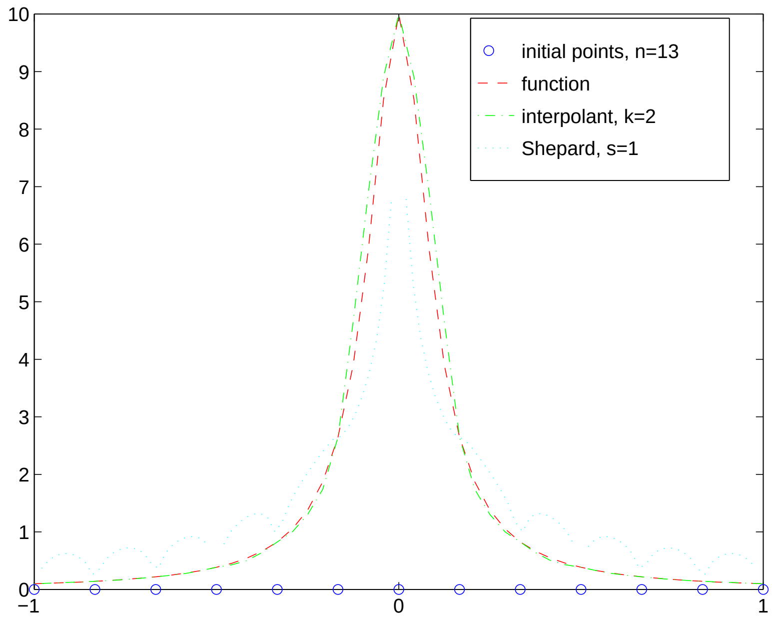

Figure 2: The function $y(x) = 1/(1 + 100\,x^2)$ on [-1,1] , our interpolant and the Shepard's one

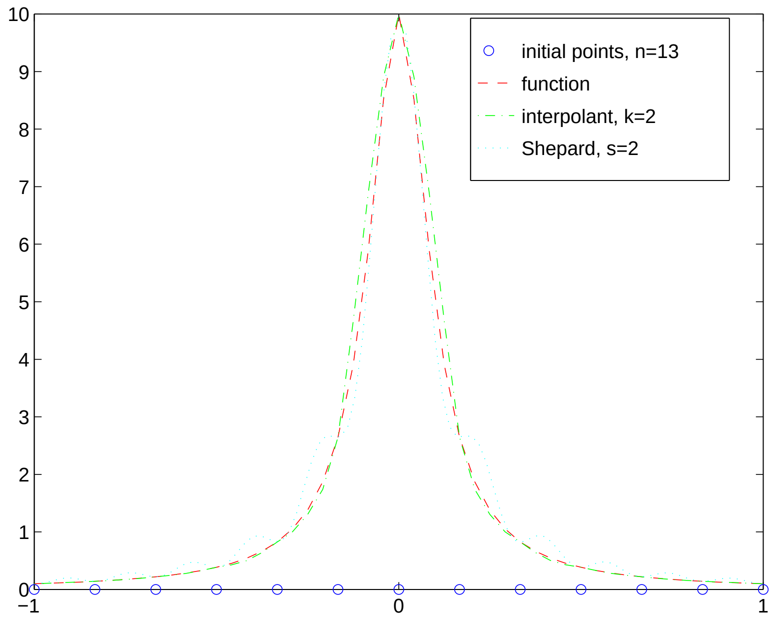

Figure 3: The function $y(x) = 1/(1 + 100\,x^2)$ on [-1,1] , our interpolant and the Shepard's one

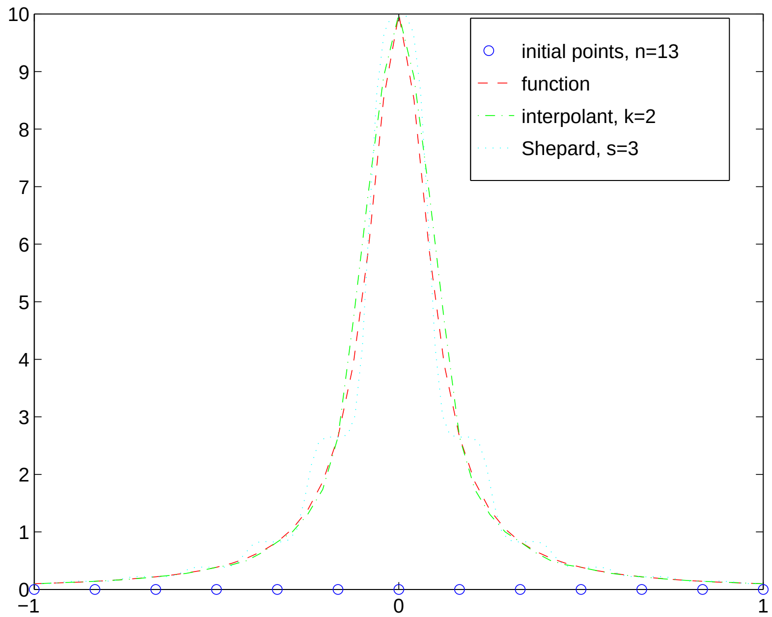

Figure 4: The function $y(x) = 1/(1 + 100\,x^2)$ on [-1,1] , our interpolant and the Shepard's one

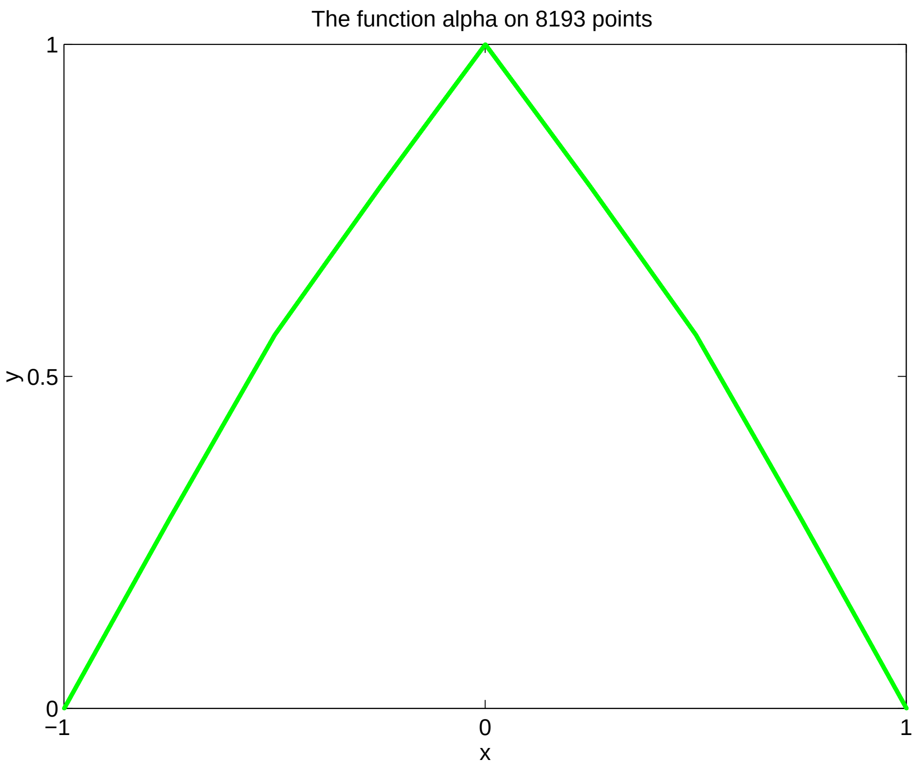

Figure 5: The function $\alpha(x)$ on $2^{13}+1$ points

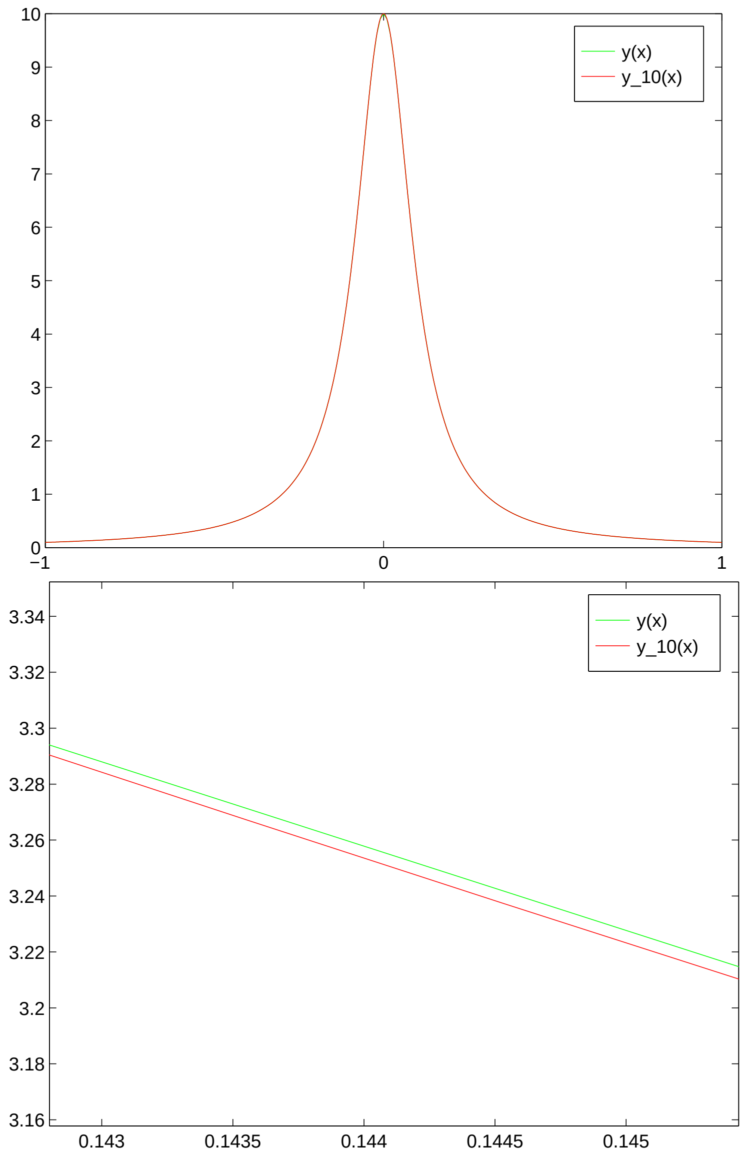

Figure 6: The function $y(x) = \frac{10}{1+100\,x^2}$ and its approximant y_{10} (above) and a zoom (below).