

REGULARITY OF IRREGULAR SUBDIVISION

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ABSTRACT. We study the smoothness of the limit function for one dimensional unequally spaced interpolating subdivision schemes. The new grid points introduced at every level can lie in irregularly spaced locations between old, adjacent grid points and not only midway as is usually the case. For the natural generalization of the four point scheme introduced by Dubuc and Dyn, Levin, and Gregory, we show that, under some geometric restrictions, the limit function is always C^1 ; under slightly stronger restrictions we show that the limit function is almost C^2 , the same regularity as in the regularly spaced case.

1. INTRODUCTION

Subdivision is a powerful mechanism for the construction of smooth curves and surfaces. The main idea behind subdivision is to iterate upsampling and local averaging to build complex geometrical shapes. Originally such schemes were studied in the context of corner cutting [12, 5] as well as for building piecewise polynomial curves, e.g., the de Casteljau algorithm for Bernstein-Bézier curves [11] or algorithms for the iterative generation of splines [24, 1]. Later subdivision was studied independently of spline functions [18, 16, 13, 2, 3, 4]. Around the same time it was noted that subdivision fits into the framework of wavelets and multiresolution analysis [26, 8].

Smoothness of spline functions follows from simple algebraic conditions on the polynomial segments at the knots. However, when the limit function of a subdivision scheme is not a spline, convergence and smoothness are usually harder to prove. Various approaches have been explored to find the Hölder exponent of the limit function [16, 18, 13, 27, 9, 10, 19, 17, 2, 30] or to determine its Sobolev class [36, 20]. Note that these references are only some of the earliest studies; both approaches have given rise to a much larger literature, outside the scope of this paper. All these results are concerned with a regular (or sometimes called uniform) grid, i.e., at each stage new grid points are introduced in the middle of two old grid points. The most common tools used are the commutation formula (by which the order of the subdivision can be reduced), the Fourier transform, and spectral analysis.

In the spline context, knot insertion algorithms early on allowed for splines with non-equally spaced knots. This extra flexibility is crucial in developing algorithms for computer aided geometric design, see e.g. [21]. Later a global subdivision scheme for non-uniform splines was introduced in [28]. Again smoothness results are relatively easy given that the analytic form of the limit function is known.

Key words and phrases. regularity, irregular samples, subdivision, interpolation, scaling function, wavelet.

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Only recently have people started working on subdivision for non-equally spaced knots. Here we distinguish two settings. The *semi-regular* case, where the original samples are non-equally spaced, but the subdivision scheme still introduces new grid points midway between old ones, and the *irregular* case, where new grid points need not be in the middle between old ones even ad infinitum. For example, the family of Lagrange interpolating schemes [14] of which the four point scheme is the cubic case, can easily be generalized to both the semi-regular case [39, 37] and the irregular case [35]. When building splines the limit functions are piecewise polynomial, and there is no need to distinguish between the semi-regular case and the irregular case: both cases are commonly referred to as non-uniform.

Almost all work on smoothness for non-equally spaced grids concerns the semi-regular case; the subdivision scheme becomes spatially variant, but it is still stationary using the same weights across levels. Consequently the Fourier transform can no longer be used, but one can still rely on spectral analysis. In [39] Warren shows how spectral analysis can be used to analyze interpolating subdivision in the semi-regular case. He shows that the four point interpolating scheme in the semi-regular case yields a C^1 limit function. Several results have been obtained in the higher dimensional semi-regular setting. The problem then again becomes harder as now also the topology can be irregular. We refer to [29, 37, 40] for more details.

In the irregular case the subdivision scheme becomes both spatially variant and non-stationary. Smoothness results are not straightforward; because the subdivision is spatially variant the Fourier transform can no longer be used and because it is non-stationary even spectral analysis cannot help. The aim of this paper is to study the regularity of limit functions of subdivision in the irregular (ad infinitum) one dimensional case. We show that the commutation formula still holds in the irregular case and use it as the main tool in our analysis. For example, with a very mild condition on the irregularity of the grid, the four point scheme converges to a C^1 function in the irregular case; we also show how the irregularity of the grid affects the fractional smoothness exponent. In fact, with a more restrictive condition on the irregular grid, the four point scheme actually converges to a $C^{2-\epsilon}$ function, the same regularity as in the regular case.

Why should one even care about the irregular setting? Is not the semi-regular setting sufficient? There the user provides the coarse level grid points and after that the subdivision might as well use the midpoints to synthesize the curve. In this setup, indeed, the semi-regular setting is sufficient to generate smooth functions. However, to have more control over the geometric shape of a curve a designer may want to insert new points at arbitrary locations independent of the underlying parameterization. Keep in mind that the interpolating subdivision schemes considered in this paper provide an editing mechanism quite different from traditional spline knot insertion. While knot insertion increases the number of knots in the *coarsest* grid, we consider the addition of new points on *finer* grids. Moreover, knot insertion initially does not affect the shape of the limit function, while in our case the limit function does depend on the location of the new points.

Another application that calls for the irregular setting is the need for wavelets and multiresolution analysis for irregular samples. Here the user provides data, sampled on a closely spaced but irregular grid, which we can think of as the finest level grid. Resampling onto a regular grid is typically costly and may generate unwanted artifacts. In [35] it is shown how to then build a multiresolution analysis and an associated wavelet transform on the original grid. The main idea is to downsample the original grid and introduce spatially variant filter banks using the

lifting scheme. Once the multiresolution is defined, wavelet based algorithms such as compression and denoising, familiar from the regular case, can be carried out in the same way in the irregular setting. The wavelet basis functions from the coarsest level are now generated with a subdivision scheme where the new points are no longer midpoints but are dictated by the finest level grid on which the data was sampled. They are no longer translates and dilates of one fixed function, but form an instance of so-called “second generation wavelets” [34]. One could now use the semi-regular setting to argue that using midpoints *beyond* the finest level leads to a smooth limit function. However, in a practical setting one often cannot afford or one does not care to synthesize functions on levels finer than the original finest level. Instead all processing is done on the original grid or the coarser grids. Given that the finest and coarsest level can be arbitrarily far apart, the irregular setting then becomes the correct model.

One may also wonder why it is necessary to spatially adapt the weights. Could not one, even in the irregular setting, stick with the fixed weights of the regular subdivision? Indeed, the irregular grid points can always be thought of as some remapping of the regular grid points. In case this remapping is a smooth function, this is correct and one can stick with the fixed weights. This corresponds to requiring that the finer the level, the more regular the grid becomes. In our setting, we shall not require this, and our constraint on the irregularity of the grids will not depend on the level. We thus allow for non-smooth remapping functions, e.g, to accommodate sudden changes in sampling density; sticking with fixed weights would then result in non-smooth limit functions. An example of this is given in Figure 1. Other reasons why adapting the weights

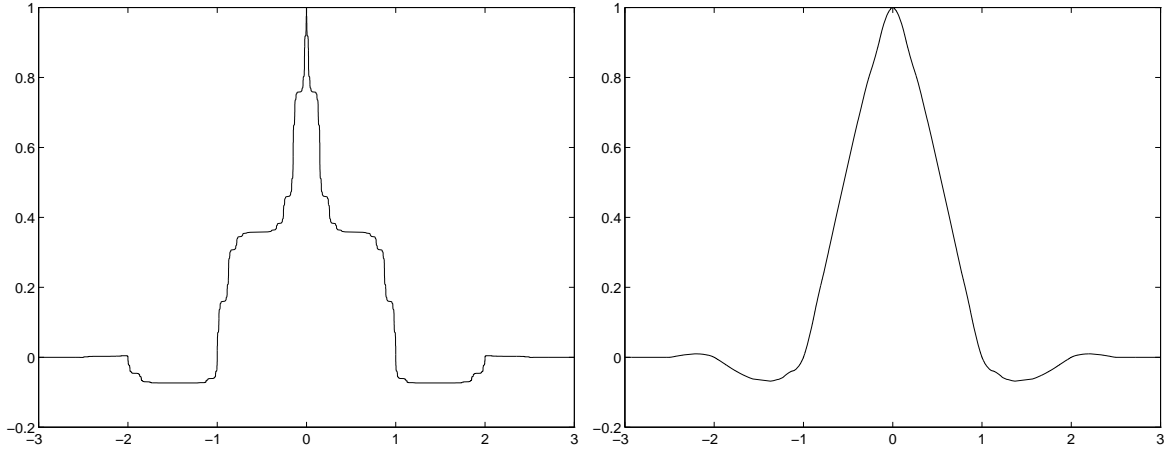


FIGURE 1. An example why remapping does not work. Left the limit function with fixed weights, right the one with weights adjusted to the geometry. The same irregular grid is used in both figures. The corresponding remapping function is shown in Figure 2.

is important in the semi-regular setting come from curve and surface generation; in [23] it is shown that adaptive weights lead to curves with much less over- and undershoot, while [41] shows that fixed weights lead to unwanted artifacts in surface generation.

Another way to motivate subdivision with varying weights comes from variational approaches, as explored by Kobbelt [22] and Warren [38]. As pointed out above, one cares about more than

simply Hölder smoothness in applications; the missing ingredient is often referred to as “fairness” in the graphics literature [31]. A smooth curve is also called fair in case it is visually pleasing and has no unwanted undulations; fairness is thus a rather subjective notion. Typically curves minimizing certain variational functionals are fair. In [22, 23, 38] it is shown how to find a subdivision scheme whose limit function optimizes a certain given functional. For irregular grids variational subdivision lead to spatially varying weights. While variation arguments for building subdivision differ from the polynomial interpolation we consider in this paper, the tools we develop are useful to study the regularity of general subdivision schemes with varying weights.

The rest of the paper is organized as follows. In Section 2 we introduce irregular multi-level grids and define quantities to measure the irregularity of the grid; we distinguish in particular homogeneous and dyadically balanced multi-level grids. Section 3 defines subdivision and Section 4 shows how to build derived subdivision schemes using the commutation formula. This is worked out in detail for the case of cubic Lagrange subdivision in Section 5. Section 6 contains the general results for homogeneous grids. Section 7 provides estimates on the growth of higher-order differences for the cubic Lagrange subdivision, and Section 8 concludes the regularity analysis of that scheme in the homogeneous case. In Section 9 we revisit the cubic Lagrange case, but now without assuming homogeneity of the grid. Section 10 discusses higher order cases while Section 11 concludes with some comments.

2. MULTI-LEVEL GRIDS

We start with a sequence of grids X_j on the real line, for $j \in \mathbf{N}$. Each grid X_j is a strictly increasing sequence of points $\{x_{j,k} \in \mathbf{R} \mid k \in \mathbf{Z}\}$. Moreover, these grids are consecutive binary refinements of the initial grid X_0 , i.e., $X_j \subset X_{j+1}$ and $x_{j+1,2k} = x_{j,k}$ for all j and k . Thus in every refinement step we insert one odd point $x_{j+1,2k+1}$ between each adjacent pair of even points $x_{j,k} = x_{j+1,2k}$ and $x_{j,k+1} = x_{j+1,2k+2}$. We will refer to j as the level of the grid point $x_{j,k}$, where $j = 0$ is the initial, coarsest level. The length of the interval between $x_{j,k}$ and $x_{j,k+1}$ is given by $d_{j,k}$:

$$d_{j,k} = x_{j,k+1} - x_{j,k}.$$

We also introduce polynomial sequences defined by

$$\pi_j^{[p]} = \{x_{j,k}^p \mid k \in \mathbf{Z}\}.$$

We impose some restrictions on the irregularity of a multi-level grid by requiring that certain characteristic numbers be bounded. The first characteristic number can be introduced for any multi-level grid:

$$\gamma = \sup_{j,k} \frac{\max(d_{j,k+1}, d_{j,k-1})}{d_{j,k}};$$

this plainly captures ratios of neighboring intervals without paying any attention to the refinement procedure. Note that one always has $\gamma \geq 1$; equality ($\gamma = 1$) corresponds to the equally spaced case. If $\gamma < \infty$, then we shall say that the multi-level grid is *homogeneous*; in Appendix A we show how this uniform bound, across scales, on the ratio between neighboring intervals is reminiscent of the definition of spaces of homogeneous type.

Another characteristic number is specific for a grid built by dyadic refinement and concerns the ratio between the lengths of any interval at level j and that of its two “children” at level $j + 1$:

$$\beta = \inf_{j,k} \frac{\min(d_{j+1,2k}, d_{j+1,2k+1})}{d_{j,k}}. \quad (1)$$

It thus provides a lower bound for the ratio of the smallest child versus the parent. Note that $0 \leq \beta \leq 1/2$, with $\beta = 1/2$ in the equally spaced and semi-regular case. We say that the multi-level grid is *dyadically balanced* if $\beta > 0$.

If a homogeneous multi-level grid is generated by a dyadic refinement, then it is dyadically balanced; since $d_{j,k} = d_{j+1,2k} + d_{j+1,2k+1}$, it follows that $\beta \geq 1/(1 + \gamma)$. However, a dyadically balanced grid is not necessarily homogeneous; an example is $x_{0,l} = l \in \mathbf{Z}$, $x_{j+1,2l+1} = x_{j,l} + d_{j,l}/3$, for which $\beta = 1/3$, but $\gamma = \infty$ since $d_{j,0} = (1/3)^j$ and $d_{j,-1} = (2/3)^j$.

We shall consider both the homogeneous case ($\gamma < \infty$) and the non-homogenous but dyadically balanced case ($\gamma = \infty$, $\beta > 0$) below. Altogether the homogeneous case is much easier for analysis; one reason is that close intervals have about the same length, so that we can define the “local scale” of the grid around any point $x_{j,k}$.

We often assume that $\inf_k d_{0,k} > 0$ and $\sup_k d_{0,k} < \infty$. Under this condition the semi-regular case is homogeneous. These are not severe restrictions. In a practical situation we start only from compactly supported initial data, i.e., only a finite number of non-zero function values. If the original grid then does not have uniform bounds on the $d_{0,k}$, we may, without loss of generality, assume such bounds. Indeed, given that we consider only local subdivision, the limit function at any given point depends only on a finite set of initial data centered around that point. Thus without uniform bounds we still have the same results on compact sets, with estimates that hold uniformly on the compact set under consideration. The only difference is that some of our constants will depend on the compact set.

Note that these uniform bounds imply that

$$d_{j,k} \leq (1 - \beta) d_{j-1, \lfloor k/2 \rfloor} \leq (1 - \beta)^j \sup_k d_{0,k} \quad (2)$$

and

$$d_{j,k} \geq \beta d_{j-1, \lfloor k/2 \rfloor} \geq \beta^j \inf_k d_{0,k}, \quad (3)$$

even if the grid is not homogenous.

In order to get an idea of how irregular these grids can be, let us consider the function $\xi(x)$ defined on the dyadics by $\xi(k 2^{-j}) = x_{j,k}$. The strictly increasing function ξ can be defined elsewhere by continuity; it is easy to see that ξ is a bijection on \mathbf{R} . This is precisely the remapping function mentioned in the introduction. Note that (2) implies that ξ is Hölder continuous with exponent $|\log_2(1 - \beta)|$ while (3) implies that ξ^{-1} is Hölder continuous with exponent $|\log_2 \beta|^{-1}$. We give some examples on $[-1, 1]$ where $x_{0,k} = k$. Figure 2 gives the remapping in a homogeneous case for $\gamma = 5.6667$; the grid in this example is generated by the rules

$$x_{j+1,4k+1} = \beta x_{j,2k} + (1 - \beta) x_{j,2k+1} \quad \text{and} \quad x_{j+1,4k+3} = (1 - \beta) x_{j,2k+1} + \beta x_{j,2k+2},$$

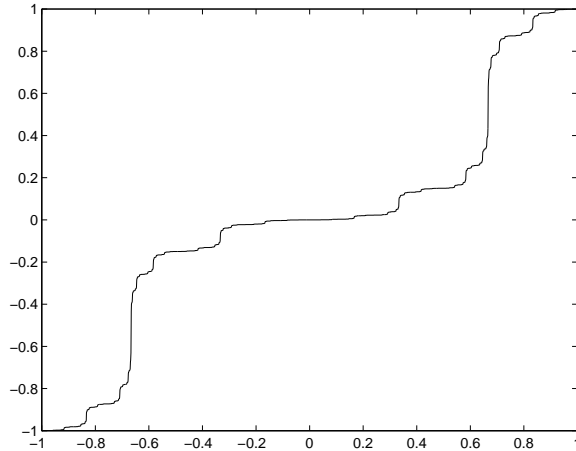


FIGURE 2. The remapping function ξ in a homogeneous case for $\gamma = 5.6667$.

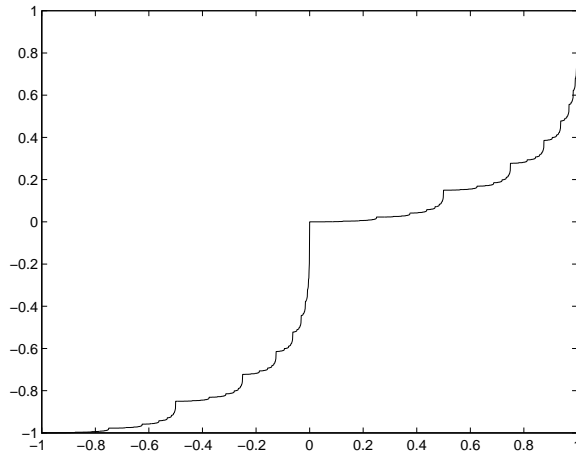


FIGURE 3. The remapping function ξ in a dyadically balanced but inhomogeneous case for $\beta = 0.15$. Note the abrupt change in sampling density around the origin.

where $\beta = 0.15$. Figure 3 gives the remapping in a dyadically balanced but inhomogeneous case with the same β ($\beta = 0.15$); in this case the grid is generated by $x_{j+1,2k+1} = \beta x_{j,k} + (1 - \beta) x_{j,k+1}$. In this case the sampling density changes abruptly around the origin.

Finally, the value of γ and β depend on the entire mesh, while strictly speaking we only need these bounds in the limit. Thus the results in this paper also hold if γ is defined using $\limsup_{j>0} \sup_k$ and β using $\liminf_{j>0} \inf_k$.

3. SUBDIVISION

3.1. Definition. Subdivision starts with a set of initial values $f_0 = \{f_{0,k}\}$ which live on the coarsest grid X_0 . Given that we mostly work with compactly supported f_0 we may assume, without loss of generality, that the sequence f_0 is bounded.

The subdivision scheme S is a sequence of linear operators S_j , $j \geq 0$, which iteratively compute values $f_j = \{f_{j,k}\}$ on the finer grids:

$$f_{j+1} = S_j f_j.$$

We assume that all S_j as well as all other operators we consider are bounded on ℓ^∞ , unless explicitly stated. Subdivision gives us values defined on the grid points $x_{j,k}$. However, the ambition is to synthesize a continuous limit function $\varphi(x)$ defined for all $x \in \mathbf{R}$ as

$$\varphi(x) = \lim_{j \rightarrow \infty} f_{j,k_j(x)},$$

where $x_{j,k_j(x)}$ is the grid point on level j to the left of x , i.e., $k_j(x) = \max\{l : x_{j,l} \leq x\}$. This paper is concerned with analyzing the existence and smoothness of $\varphi(x)$.

The subdivision operator S_j can be viewed as a matrix with the subdivision coefficients $S_{j,l,k}$ as its entries; the coefficients are defined by

$$f_{j+1,l} = \sum_k S_{j,l,k} f_{j,k}.$$

For practical reasons it is important that the above sum is finite. We say that a subdivision scheme is *local* in case, for some $B \in \mathbf{N}$,

$$S_{j,l,k} = 0 \quad \text{for} \quad |l - 2k| > B.$$

A new value at the point $x_{j+1,l}$ can then be found by the finite sum

$$f_{j+1,l} = \sum_{k=\lceil \frac{l-B}{2} \rceil}^{\lfloor \frac{l+B}{2} \rfloor} S_{j,l,k} f_{j,k}.$$

In this paper we consider only local subdivision schemes; these are automatically bounded in ℓ^∞ if $\sup_{j,l,k} |S_{j,l,k}| < \infty$.

3.2. Interpolating subdivision. We say that a subdivision scheme is *interpolating* if in each subdivision step the values at the even grid points are kept, i.e.

$$f_{j+1,2k} = f_{j,k} \quad \text{for all } j \text{ and } k.$$

The limiting function thus interpolates the original data

$$\varphi(x_{0,k}) = f_{0,k}.$$

As a particular case we consider Lagrange interpolating subdivision on irregular grids used in [39, 35], which can be thought of as a generalization of the subdivision schemes in [16, 18, 13, 14].

To find the value $f_{j+1,2k+1}$ at an odd grid point, this scheme uses $P(x)$, the interpolating polynomial of degree $2W - 1$ determined by the values at $2W$ neighboring even grid points,

$$P(x_{j,k+u}) = f_{j,k+u} \quad \text{for} \quad -W + 1 \leq u \leq W.$$

The new value $f_{j+1,2k+1}$ is then simply the evaluation of $P(x)$ at the odd grid point,

$$f_{j+1,2k+1} = P(x_{j+1,2k+1}).$$

It is easy to verify that this scheme is local with $B = 2W - 1$. In case $W = 1$ it amounts to linear interpolation: the limit function is simply a piecewise linear function interpolating the

coarsest level values. In this paper we will focus on the cubic ($W = 2$) case, which is illustrated in Figure 4.

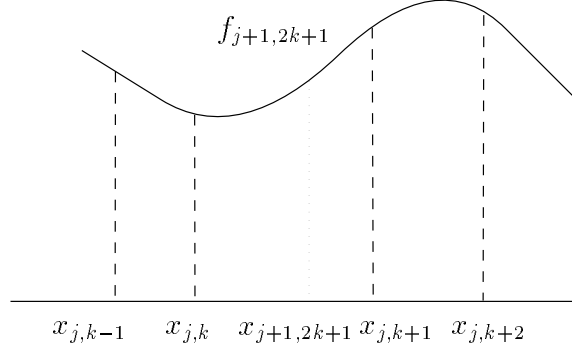


FIGURE 4. Cubic interpolation: The value $f_{j+1,2k+1}$ at the odd gridpoint $x_{j+1,2k+1}$ is obtained by evaluating a cubic polynomial $P(x)$ interpolating values at 4 neighboring even gridpoints $x_{j+1,2k-2} = x_{j,k-1}, \dots, x_{j+1,2k+4} = x_{j,k+2}$.

The coefficients of the subdivision scheme are given by Lagrange interpolation:

$$S_{j,2k+1,k+u} = \prod_{\substack{-W < v \leq W \\ v \neq u}} \frac{x_{j+1,2k+1} - x_{j,k+v}}{x_{j,k+u} - x_{j,k+v}}. \quad (4)$$

Because this is an interpolating scheme, the even rows are given by

$$S_{j,2k,k+u} = \delta_u.$$

By definition, the interpolating subdivision scheme exactly reproduces polynomials up to degree $2W - 1$,

$$\sum_k S_{j,l,k} x_{j,k}^p = x_{j+1,l}^p \quad \text{for } 0 \leq p < 2W,$$

or, using the notation for polynomial sequences,

$$S_j \pi_j^{[p]} = \pi_{j+1}^{[p]} \quad \text{for } 0 \leq p < 2W. \quad (5)$$

4. DERIVED SUBDIVISION SCHEMES

Because the Lagrange interpolating subdivision scheme preserves polynomials up to a certain order, it has a lot of structure which we can exploit to prove convergence and regularity. This is done through a commutation formula which relates subdivision and finite differences. The commutation formula for the regular setting is described in [25]; here we show how it works in the irregular interpolating setting. This commutation formula will be the main ingredient for our analysis; consequently our techniques can be applied to other subdivision schemes than the cubic Lagrange interpolation that is our main example, as long as they satisfy the commutation formula. We first introduce some notation.

4.1. Finite differences. We first introduce the standard forward divided differences for values $f_{j,k}$ defined on grid points $x_{j,k}$ for fixed j [33]. The zeroth order divided difference is simply the function value:

$$[x_{j,k}]f = f_{j,k}.$$

The higher order divided differences are defined recursively:

$$[x_{j,k}, \dots, x_{j,k+p}]f = \frac{[x_{j,k+1}, \dots, x_{j,k+p}]f - [x_{j,k}, \dots, x_{j,k+p-1}]f}{x_{j,k+p} - x_{j,k}}.$$

We let $f_j^{[p]}$ be the set $\{f_{j,k}^{[p]} \mid k \in \mathbf{Z}\}$, where

$$f_{j,k}^{[p]} = [x_{j,k}, \dots, x_{j,k+p}]f,$$

and introduce the shorthand

$$d_{j,k}^{[p]} = x_{j,k+p} - x_{j,k},$$

so that

$$f_{j,k}^{[p]} = \frac{f_{j,k+1}^{[p-1]} - f_{j,k}^{[p-1]}}{d_{j,k}^{[p]}}. \quad (6)$$

The sequences $f_j^{[p]}$ are related through the difference operator $D_j^{[p]}$:

$$f_j^{[p]} = D_j^{[p]} f_j^{[p-1]}.$$

We can think of $D_j^{[p]}$ as an infinite matrix with elements $D_{j,k,l}^{[p]}$,

$$f_{j,k}^{[p]} = \sum_l D_{j,k,l}^{[p]} f_{j,l}^{[p-1]},$$

where

$$D_{j,k,l}^{[p]} = \begin{cases} -1/d_{j,k}^{[p]} & \text{if } l = k, \\ 1/d_{j,k}^{[p]} & \text{if } l = k + 1, \\ 0 & \text{otherwise.} \end{cases}$$

We define the higher order difference operator $\Delta_j^{[p]}$ as

$$\Delta_j^{[p]} = D_j^{[p]} D_j^{[p-1]} \dots D_j^{[1]},$$

so that $f_j^{[p]} = \Delta_j^{[p]} f_j^{[0]}$. Is it easy to verify that $\Delta_j^{[p]}$ applied to the polynomial sequence of degree p gives a constant sequence. More precisely [33]

$$\Delta_j^{[p]} \pi_j^{[p]} = \pi_j^{[0]}. \quad (7)$$

4.2. **Commutation formula.** Given a subdivision scheme for function values

$$f_{j+1}^{[0]} = S_j^{[0]} f_j^{[0]},$$

we want to define, if possible, a local subdivision scheme for the divided differences,

$$f_{j+1}^{[1]} = S_j^{[1]} f_j^{[1]}.$$

We study the condition under which such a derived subdivision scheme exists. Start out with a local subdivision scheme $S^{[0]}$ which preserves constants, i.e., for all j and l

$$\sum_k S_{j,l,k}^{[0]} = 1.$$

In the regular setting it is well known that preservation of constants is a necessary condition for the subdivision scheme to converge. In the irregular setting this is no longer the case. However, preservation of constants is still a very natural condition. It implies that the corresponding curve subdivision scheme is coordinate independent. Without coordinate independence, curve subdivision schemes are very awkward to deal with.

We can now compute the $f_j^{[1]}$ via a local subdivision scheme:

$$\begin{aligned} f_{j+1,l}^{[1]} &= \frac{f_{j+1,l+1} - f_{j+1,l}}{d_{j+1,l}} \\ &= \frac{1}{d_{j+1,l}} \sum_{k=\lceil \frac{l-B}{2} \rceil}^{\lfloor \frac{l+B+1}{2} \rfloor} (S_{j,l+1,k} - S_{j,l,k}) f_{j,k} \\ &= \frac{1}{d_{j+1,l}} \sum_{k=\lceil \frac{l-B}{2} \rceil + 1}^{\lfloor \frac{l+B+1}{2} \rfloor} (S_{j,l+1,k} - S_{j,l,k}) (f_{j,k} - f_{j,\lceil \frac{l-B}{2} \rceil}) \quad (\text{constant reproduction}) \\ &= \frac{1}{d_{j+1,l}} \sum_{k=\lceil \frac{l-B}{2} \rceil + 1}^{\lfloor \frac{l+B+1}{2} \rfloor} (S_{j,l+1,k} - S_{j,l,k}) \sum_{m=\lceil \frac{l-B}{2} \rceil}^{k-1} d_{j,m} f_{j,m}^{[1]} \\ &= \sum_{m=\lceil \frac{l-B}{2} \rceil}^{\lfloor \frac{l+B-1}{2} \rfloor} S_{j,l,m}^{[1]} f_{j,m}^{[1]}, \end{aligned}$$

where

$$S_{j,l,m}^{[1]} = \frac{d_{j,m}}{d_{j+1,l}} \sum_{k=m+1}^{\lfloor \frac{l+B+1}{2} \rfloor} (S_{j,l+1,k} - S_{j,l,k}),$$

with the convention that $\sum_{l=l_1}^{l_2}$ is zero if $l_2 < l_1$.

Now we can repeat this process. If $S^{[1]}$ also reproduces constants we can use the same construction to define $S^{[2]}$ and so on. In general $S_j^{[p]}$ is defined as

$$S_{j,l,m}^{[p]} = \frac{d_{j,m}^{[p]}}{d_{j+1,l}^{[p]}} \sum_{k>m} (S_{j,l+1,k}^{[p-1]} - S_{j,l,k}^{[p-1]}), \quad (8)$$

as long as $S^{[p-1]}$ reproduces constants. Eventually for some q , $S^{[q]}$ will no longer reproduce constants. We then say that the *order* of the subdivision scheme $S^{[0]}$ is q . The derived subdivision schemes $S^{[p]}$, with $0 < p \leq q$, are all local; observe from (8) that if $S_{j,l,k}^{[p-1]} = 0$ for $|2l - k| > B$ then the same is true for $S_{j,l,k}^{[p]}$. A more careful inspection shows that every application of commutation (if possible) will reduce the column width W by one, so that $B = \lceil (W - 1)/2 \rceil$ is reduced by one after every other application of commutation; see Section 5 for examples.

Subtracting equation (8) for $S_{j,l,m-1}^{[p]}$ and $S_{j,l,m}^{[p]}$ also implies that

$$\frac{S_{j,l,m-1}^{[p]}}{d_{j,m-1}^{[p]}} - \frac{S_{j,l,m}^{[p]}}{d_{j,m}^{[p]}} = \frac{S_{j,l+1,m}^{[p-1]} - S_{j,l,m}^{[p-1]}}{d_{j+1,l}^{[p]}}, \quad (9)$$

The derived subdivision schemes thus satisfy the *commutation formula*

$$S_j^{[p]} D_j^{[p]} = D_{j+1}^{[p]} S_j^{[p-1]} \quad \text{for } 0 < p \leq q. \quad (10)$$

Combining these equations leads to

$$S_j^{[p]} \Delta_j^{[p]} = \Delta_{j+1}^{[p]} S_j^{[0]} \quad \text{for } 0 < p \leq q. \quad (11)$$

Remark: Note that our definition of order does not imply that for $0 < p \leq q$ the order of $S^{[p]}$ is $q - p$. This “unnatural” feature follows from the fact that our present definitions are geared towards interpolating subdivision schemes. The derived schemes obtained from an interpolating scheme are not themselves interpolating, and the simple definition of “order” given here does not apply to them; this is also the reason why the difference operators $D^{[p]}$ for $p > 1$ differ from $D^{[1]}$. For a general subdivision scheme, it turns out [?] that the definition of the appropriate difference operator, leading to a satisfactory derived scheme, depends on the data sequences that through subdivision lead to polynomial limit functions. With the more careful definition of difference operators and order of a (not necessarily interpolating) subdivision scheme given in [?], the q -th derived scheme of an initial (interpolating) scheme of order p turns out to be indeed of order $p - q$.

4.3. Commutation in the other direction. It is also possible to use the commutation formula in the other direction, i.e., to start with a local subdivision scheme $S^{[p]}$ and construct a scheme $S^{[p-1]}$ for lower order differences. It is well known that for shift-invariant schemes this can always be done; it simply involves an extra factor $(1 + z)/2$ in the symbol of the scheme and $S^{[p-1]}$ thus always is local. In the irregular case the situation becomes more complex and a condition on $S^{[p]}$ must be satisfied to guarantee that $S^{[p-1]}$ is local.

This can be seen as follows. We start with a compactly supported sequence $f_j^{[p-1]}$ and want to compute $f_{j+1}^{[p-1]}$ and check that it too is compactly supported, as can easily be seen from (6).

We first apply differencing to find $f_j^{[p]}$ and then use $S_j^{[p]}$ to find $f_{j+1}^{[p]}$. Given that $S^{[p]}$ is local, $f_{j+1}^{[p]}$ is compactly supported; for $f_{j+1}^{[p-1]}$ to be compactly supported, we need that

$$\sum_l d_{j+1,l}^{[p]} f_{j+1,l}^{[p]} = 0.$$

The left hand side is equal to (all summations finite)

$$\begin{aligned} \sum_l d_{j+1,l}^{[p]} \sum_k S_{j,l,k}^{[p]} f_{j,k}^{[p]} &= \sum_l \sum_k d_{j+1,l}^{[p]} S_{j,l,k}^{[p]} \frac{f_{j,k+1}^{[p-1]} - f_{j,k}^{[p-1]}}{d_{j,k}^{[p]}} \\ &= \sum_k \frac{f_{j,k+1}^{[p-1]} - f_{j,k}^{[p-1]}}{d_{j,k}^{[p]}} \sum_l d_{j+1,l}^{[p]} S_{j,l,k}^{[p]}. \end{aligned}$$

This is zero if and only if

$$\sum_l d_{j+1,l}^{[p]} S_{j,l,m}^{[p]} = C_j d_{j,m}^{[p]},$$

with C_j independent of m ; this give us a necessary and sufficient condition for $S^{[p]}$ to be the derived subdivision of $S^{[p-1]}$. Now it is also clear why there is no extra condition in the shift-invariant case; because all columns of $S^{[p]}$ are shifted copies of each other the above condition is automatically satisfied. In Appendix B we show that the constant C_j is equal to one (provided that the subdivision scheme preserve constants, which we shall always assume).

It now follows from (9) that

$$S_{j,l,m}^{[p-1]} = \sum_{n=l}^{2m+B} \left(\frac{S_{j,n,m}^{[p]}}{d_{j,m}^{[p]}} - \frac{S_{j,n,m-1}^{[p]}}{d_{j,m-1}^{[p]}} \right) d_{j+1,n}^{[p]}, \quad (12)$$

assuming that $S_{j,l,k}^{[p]} = 0$ for $|l - 2k| > B$.

Equations (8) and (12) show that for a homogeneous multi-level grid the $S_j^{[p]}$ as operators on ℓ^∞ are bounded uniformly in j if and only if the $S_j^{[p-1]}$ are. For a merely dyadically balanced grid this is not clear a priori.

We denote by $\varphi^{[p]}(x)$ the limit function of a derived subdivision scheme $S^{[p]}$ (if this limit exists):

$$\varphi^{[p]}(x) = \lim_{j \rightarrow \infty} f_{j,k_j}^{[p]}(x). \quad (13)$$

Later we will study how the $\varphi^{[p]}(x)$ for different p are connected.

4.4. The order of polynomial interpolating subdivision. Let us now find the order of an interpolating subdivision scheme $S^{[0]}$ which reproduces polynomials up to degree V :

$$S_j^{[0]} \pi_j^{[p]} = \pi_{j+1}^{[p]} \quad \text{for } 0 \leq p < V.$$

Note that $S^{[0]}$ does not have to be the Lagrange interpolating scheme where $V = 2W$, it only has to be interpolating and reproduce polynomial sequences.

We will reason by induction on p . Start out assuming that $S^{[p-1]}$ reproduces constants and $p < V$. In the previous section we have shown that we can then define a local scheme $S^{[p]}$ that satisfies

$$S_j^{[p]} \Delta_j^{[p]} = \Delta_{j+1}^{[p]} S_j^{[0]}.$$

Consequently

$$S_j^{[p]} \Delta_j^{[p]} \pi_j^{[p]} = \Delta_{j+1}^{[p]} S_j^{[0]} \pi_j^{[p]}.$$

Using (7) the left hand side is equal to

$$S_j^{[p]} \Delta_j^{[p]} \pi_j^{[p]} = S_j^{[p]} \pi_j^{[0]}.$$

Given that $p < V$, polynomial reproduction gives that the right hand side is equal to

$$\Delta_{j+1}^{[p]} S_j^{[0]} \pi_j^{[p]} = \Delta_{j+1}^{[p]} \pi_{j+1}^{[p]} = \pi_{j+1}^{[0]}.$$

Combined this shows that $S^{[p]}$ reproduces constants:

$$S_j^{[p]} \pi_j^{[0]} = \pi_{j+1}^{[0]}.$$

We can now start the induction with $p = 1$ and let it run till $p = V - 1$. It then immediately follows that the order of the interpolating subdivision $S^{[0]}$ is V .

The derived subdivision schemes are essential in determining the regularity of $S^{[0]}$. The schemes for the higher order differences become easier to analyze because they are “narrower”, and their decay or growth can be used to estimate the regularity of the limit function. (See the example of cubic Lagrange interpolation below.)

5. SUBDIVISION SCHEMES DERIVED FROM CUBIC LAGRANGE INTERPOLATING SUBDIVISION

Using Maple, an algebraic manipulation package, we computed the 4 derived subdivision schemes starting from the cubic interpolating scheme ($W = 2$). We start with $S^{[0]}$ where each even row is a Dirac sequence and each odd row has 4 non-zero entries. The matrix has the following structure:

$$\begin{array}{cccccccc}
 2k - 3 & \left[\begin{array}{ccccccc}
 * & * & * & * & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & * & * & * & * & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & \times & \times & \times & \times & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & * & * & * & *
 \end{array} \right] \\
 k - 3 & k - 2 & k - 1 & k & k + 1 & k + 2 & k + 3
 \end{array}$$

Here and below * stands for a non-zero entry, while the \times stands for a non-zero entry given explicitly in the text. The coefficients on row $2k + 1$ follow immediately from (4)

$$\begin{aligned}
S_{j,2k+1,k-1}^{[0]} &= -\frac{d_{j+1,2k}^{[1]} d_{j+1,2k+1}^{[1]} d_{j+1,2k+1}^{[3]}}{d_{j+1,2k-2}^{[2]} d_{j+1,2k-2}^{[4]} d_{j+1,2k-2}^{[6]}} \\
S_{j,2k+1,k}^{[0]} &= \frac{d_{j+1,2k-2}^{[3]} d_{j+1,2k+1}^{[1]} d_{j+1,2k+1}^{[3]}}{d_{j+1,2k-2}^{[2]} d_{j+1,2k}^{[2]} d_{j+1,2k}^{[4]}} \\
S_{j,2k+1,k+1}^{[0]} &= \frac{d_{j+1,2k-2}^{[3]} d_{j+1,2k}^{[1]} d_{j+1,2k+1}^{[3]}}{d_{j+1,2k-2}^{[4]} d_{j+1,2k}^{[2]} d_{j+1,2k+2}^{[2]}} \\
S_{j,2k+1,k+2}^{[0]} &= -\frac{d_{j+1,2k-2}^{[3]} d_{j+1,2k}^{[1]} d_{j+1,2k+1}^{[1]}}{d_{j+1,2k-2}^{[6]} d_{j+1,2k}^{[4]} d_{j+1,2k+2}^{[2]}}.
\end{aligned}$$

In the equally spaced case this becomes the well known four point scheme which uses the stencil $\{-1/16, 9/16, 9/16, -1/16\}$ [18, 14].

For $S_j^{[1]}$ each row contains only three non-zero elements. The even row at $2k - 2$ and the odd row at $2k - 1$ have only entries in the columns $k - 2$, $k - 1$, and k . The matrix thus has the following structure:

$$\begin{array}{r}
2k - 3 \\
2k - 2 \\
2k - 1 \\
2k \\
2k + 1 \\
2k + 2
\end{array}
\begin{bmatrix}
* & * & * & 0 & 0 & 0 \\
0 & \times & \times & \times & 0 & 0 \\
0 & \times & \times & \times & 0 & 0 \\
0 & 0 & * & * & * & 0 \\
0 & 0 & * & * & * & 0 \\
0 & 0 & 0 & * & * & *
\end{bmatrix}$$

$k - 3 \quad k - 2 \quad k - 1 \quad k \quad k + 1 \quad k + 2$

The $k - 2$ and k columns are easy and given by

$$\begin{aligned}
S_{j,2k-2,k-2}^{[1]} &= \frac{d_{j+1,2k-1}^{[1]} d_{j+1,2k-1}^{[3]}}{d_{j+1,2k-4}^{[4]} d_{j+1,2k-4}^{[6]}} & S_{j,2k-2,k}^{[1]} &= -\frac{d_{j+1,2k-1}^{[1]} d_{j+1,2k-4}^{[3]}}{d_{j+1,2k-2}^{[4]} d_{j+1,2k-4}^{[6]}} \\
S_{j,2k-1,k-2}^{[1]} &= -\frac{d_{j+1,2k-2}^{[1]} d_{j+1,2k-1}^{[3]}}{d_{j+1,2k-4}^{[4]} d_{j+1,2k-4}^{[6]}} & S_{j,2k-1,k}^{[1]} &= \frac{d_{j+1,2k-2}^{[1]} d_{j+1,2k-4}^{[3]}}{d_{j+1,2k-2}^{[4]} d_{j+1,2k-4}^{[6]}}.
\end{aligned}$$

The $k - 1$ column entries follow from the fact that each row sums to one:

$$S_{j,2k-2,k-1}^{[1]} = 1 - S_{j+1,2k-2,k-2}^{[1]} - S_{j+1,2k-2,k}^{[1]}$$

and

$$S_{j,2k-1,k-1}^{[1]} = 1 - S_{j+1,2k-1,k-2}^{[1]} - S_{j+1,2k-1,k}^{[1]}.$$

In the equally spaced case, this leads to the $\{1/8, 1, -1/8\}$ and $\{-1/8, 1, 1/8\}$ sequences from [15, 6].

The matrix $S_j^{[2]}$ has the following structure:

$$\begin{array}{r} 2k-3 \\ 2k-2 \\ 2k-1 \\ 2k \\ 2k+1 \end{array} \begin{bmatrix} * & * & * & 0 & 0 \\ 0 & \times & \times & 0 & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & * \end{bmatrix}$$

$$k-3 \quad k-2 \quad k-1 \quad k \quad k+1$$

The even row at $2k-2$ has 2 non-zero elements at columns $k-2$ and $k-1$:

$$S_{j,2k-2,k-2}^{[2]} = \frac{d_{j+1,2k-1}^{[3]}}{d_{2k-4}^{[6]}} \quad \text{and} \quad S_{j,2k-2,k-1}^{[2]} = \frac{d_{j+1,2k-4}^{[3]}}{d_{2k-4}^{[6]}}.$$

The odd row at $2k-1$ contains only 3 non-zero elements at columns $k-2$, $k-1$, and k . The $k-2$ and k column coefficients are given by

$$S_{j,2k-1,k-2}^{[2]} = -\frac{d_{j+1,2k-1}^{[3]} d_{j+1,2k-2}^{[1]}}{d_{j+1,2k-1}^{[2]} d_{j+1,2k-4}^{[6]}} \quad \text{and} \quad S_{j,2k-1,k}^{[2]} = -\frac{d_{j+1,2k-2}^{[3]} d_{j+1,2k+1}^{[1]}}{d_{j+1,2k-1}^{[2]} d_{j+1,2k-2}^{[6]}}$$

while the $k-1$ column coefficients follow from the fact that each row sums to one:

$$S_{j,2k-1,k-1}^{[2]} = 1 - S_{j,2k-1,k-2}^{[2]} - S_{j,2k-1,k}^{[2]}.$$

In the equally spaced case this leads to the sequences $\{1/2, 1/2\}$ and $\{-1/4, 3/2, -1/4\}$ from [6].

The subdivision scheme $S^{[3]}$ for the third difference is very simple. Each row contains only two non-zero elements. The even row at $2k-2$ and the odd row at $2k-1$ have entries only in the columns $k-2$ and $k-1$:

$$\begin{array}{r} 2k-3 \\ 2k-2 \\ 2k-1 \\ 2k \end{array} \begin{bmatrix} * & * & 0 & 0 \\ 0 & \times & \times & 0 \\ 0 & \times & \times & 0 \\ 0 & 0 & * & * \end{bmatrix}$$

$$k-3 \quad k-2 \quad k-1 \quad k$$

The coefficients are given by

$$S_{j,2k-2,k-2}^{[3]} = d_{j+1,2k-1}^{[3]} / d_{j+1,2k-1}^{[2]} \quad \text{and} \quad S_{j,2k-2,k-1}^{[3]} = -d_{j+1,2k+1}^{[1]} / d_{j+1,2k-1}^{[2]}$$

for the even row, and

$$S_{j,2k-1,k-2}^{[3]} = -d_{j+1,2k-2}^{[1]} / d_{j+1,2k-1}^{[2]} \quad \text{and} \quad S_{j,2k-1,k-1}^{[3]} = d_{j+1,2k-2}^{[3]} / d_{j+1,2k-1}^{[2]}$$

for the odd row, which in the equally spaced case leads to the $\{3/2, -1/2\}$ and $\{-1/2, 3/2\}$ sequences from [6].

The subdivision scheme $S^{[4]}$ is even more “narrow” than $S^{[3]}$, in the sense that one row out of two now has just one non-zero element; the other rows still have two. The explicit expressions for the $S_{j,l,k}^{[4]}$ are more complicated however. Fortunately, it turns out that if we consider the scheme for the *differences* (as opposed to *divided differences*) of third order divided differences, the scheme is simple again. We thus define the differences of divided differences as

$$g_{j,k}^{[p]} = f_{j,k+1}^{[p-1]} - f_{j,k}^{[p-1]} = d_{j,k}^{[p]} f_{j,k}^{[p]},$$

and consider the scheme $T^{[4]}$ so that

$$g_{j+1}^{[4]} = T_j^{[4]} g_j^{[4]}. \quad (14)$$

The $T_j^{[4]}$ matrix has the following structure:

$$\begin{array}{ccc} 2k-3 & & \\ 2k-2 & & \\ 2k-1 & & \end{array} \begin{bmatrix} * & * & 0 \\ 0 & \times & 0 \\ 0 & \times & \times \end{bmatrix}$$

$$\begin{array}{ccc} k-3 & k-2 & k-1 \end{array}$$

The even row has only one element

$$T_{j,2k-2,k-2}^{[4]} = \frac{d_{j+1,2k-2}^{[4]}}{d_{j+1,2k-1}^{[2]}}, \quad (15)$$

while the odd row has two elements

$$T_{j,2k-1,k-2}^{[4]} = -\frac{d_{j+1,2k-2}^{[1]}}{d_{j+1,2k-1}^{[2]}} \quad \text{and} \quad T_{j,2k-1,k-1}^{[4]} = -\frac{d_{j+1,2k+3}^{[1]}}{d_{j+1,2k+1}^{[2]}}. \quad (16)$$

One disadvantage of the forward difference notation is that while all the entries of the above matrices are situated near the point $x_{j,k} = x_{j+1,2k}$, their indices shift more and more to the left for higher order schemes. In some of the later sections we therefore work with central differences and we also introduce different notation for the $d_{j,k}^{[p]}$. In the homogeneous case, one can easily check the matrix entries and conclude that the $S^{[0]}$, $S^{[1]}$, $S^{[2]}$, $S^{[3]}$, and $T^{[4]}$ are bounded uniformly in j . For the inhomogeneous but dyadically balanced case, one has to look a little closer, but one still has uniform bounds for the coefficients in $S^{[1]}$, $S^{[2]}$, $S^{[3]}$, and $T^{[4]}$, as shown below. The scheme $S^{[0]}$ in the inhomogeneous case requires even more attention, for its coefficients can become unbounded. We shall come back to this in Section 9.2.

Now, consider the coefficients of the scheme $S^{[1]}$. Using symmetry it is sufficient to look at two coefficients:

$$S_{j+1,2k-2,k-2}^{[1]} = \frac{d_{j+1,2k-1}^{[1]} d_{j+1,2k-1}^{[3]}}{d_{j+1,2k-4}^{[4]} d_{j+1,2k-4}^{[6]}}$$

$$S_{j+1,2k-2,k}^{[1]} = -\frac{d_{j+1,2k-1}^{[1]} d_{j+1,2k-4}^{[3]}}{d_{j+1,2k-2}^{[4]} d_{j+1,2k-4}^{[6]}}$$

Considering the relative position of the ends of corresponding intervals it is easy to see that $d_{j+1,2k-1}^{[1]} < d_{j+1,2k-4}^{[4]}$, and $d_{j+1,2k-1}^{[3]} < d_{j+1,2k-4}^{[6]}$. In the same way one has $d_{j+1,2k-1}^{[1]} < d_{j+1,2k-2}^{[4]}$, and $d_{j+1,2k-4}^{[3]} < d_{j+1,2k-4}^{[6]}$. It follows that $|S_{j+1,2k-2,k-2}^{[1]}| < 1$, and $|S_{j+1,2k-2,k}^{[1]}| < 1$. Thus, $S^{[1]}$ has uniformly bounded coefficients.

The analogous consideration of the entries in $S^{[2]}$, $S^{[3]}$, and $T^{[4]}$ involves just one more trick. For example, consider the coefficient $S_{j,2k-2,k-1}^{[3]}$; we have

$$\left| S_{j,2k-2,k-1}^{[3]} \right| = \frac{d_{j+1,2k+1}^{[1]}}{d_{j+1,2k-1}^{[2]}} < \frac{d_{j+1,2k+1}^{[1]}}{d_{j+1,2k}^{[1]}} \leq \frac{1-\beta}{\beta},$$

where we used the fact that our multi-level grid is dyadically balanced.

6. REGULARITY ESTIMATES IN THE HOMOGENEOUS CASE

In the case of shift-invariant subdivision, the order of the scheme can be used to reduce the complexity of the regularity analysis. For the equally spaced cubic interpolation of [16, 13, 18], for instance, a combined spectral analysis of two 6×6 matrices can be reduced to the study of two 2×2 matrices; even though these reduced matrices no longer have uniformly bounded products, the rate of growth of these products (as a function of the number of factors) can be used to establish the rate of convergence of the original scheme, as well as the regularity of the limit function [9, 27]. Our goal here is to do something similar in the irregularly spaced homogeneous case: we translate bounds on the rate of growth for high-order divided differences into bounds on lower-order divided differences. Coupled with an estimate on the high-order divided differences, this leads to regularity estimates. Throughout this section we assume $\gamma < \infty$; we also assume that the subdivision scheme under consideration is of sufficiently high order, so that all the derived subdivision schemes are well-defined.

As pointed out earlier, the definition of derived subdivision schemes in this paper is particularly geared towards interpolating subdivision. However, our strategy will, *mutatis mutandis*, also work for other subdivision schemes. Therefore we keep our analysis fairly general and abstract until the end of this section, where we turn to the concrete example of cubic Lagrange interpolation.

The strategy has two parts. On the one hand, we have to prove an estimate that controls the growth through scale of the highest order divided differences that we consider. Typically, such an estimate takes the form

$$\left| f_{j,k}^{[p]} \right| \leq C \frac{\alpha^j}{(d_{j,k})^r}. \quad (17)$$

The presence of a factor of the form $(d_{j,k})^{-r}$ here expresses the dependence of the bound on the spacings in the multi-level grid. The extra factor α^j controls exponential decay (or growth) independent of the grids. In the uniformly spaced case, $d_{j,k} = 2^{-j}$ and both factors collapse into one single exponential. The proof of (17) depends strongly on the scheme under consideration, and therefore has to be constructed case-by-case. For instance, in the case of cubic Lagrange interpolation, the starting estimate (17) takes the form

$$\left| f_{j,k}^{[4]} \right| \leq C \frac{(1-\beta)^j}{d_{j,k}^3}, \quad (18)$$

with β as defined by (1) above; we shall come back below on the proof of (18); see Section 8. In the special case of uniformly spaced cubic Lagrange interpolation, $\beta = 1/2$ and $d_{j,k} = 2^{-j}$, so that (18) reduces to

$$\left| f_{j,k}^{[4]} \right| \leq C 2^{2j} \quad (19)$$

Note that we have some freedom in the choice of α and r . For instance, in the uniformly spaced cubic Lagrange interpolation, the three choices (among many others) $\alpha = 1, r = 2$; $\alpha = 2, r = 1$; $\alpha = 4, r = 0$ all reduce to (19).

The second part of the strategy is a reduction procedure that takes bounds of the type (17) and lifts them into bounds for the lower-order divided differences. Typically (17) will imply that the $f_j^{[p]}$ diverge exponentially in j , but the bound on their growth leads to a better bound on lower-order divided differences; we repeat the reduction procedure until the $f_{j,k_j(x)}^{[p]}$ become Cauchy sequences as $j \rightarrow \infty$. At that point, one can use the bounds to show convergence and regularity; see below.

The reduction procedure itself consists of three stages, repeated cyclically as often as needed. In the first stage bounds on $f_{j,k}^{[p]}$ are translated into bounds on the differences $f_{j,k+s}^{[p-1]} - f_{j,k}^{[p-1]}$; in the second stage these intra-level bounds lead to inter-level bounds on $f_{j+1,2k+t}^{[p-1]} - f_{j,k}^{[p-1]}$; in the third stage these inter-level bounds are gathered across scale into a bound on the $f_{j,k}^{[p-1]}$. The cycle can then start over again.

In summary, we start with a higher order scheme which diverges but which is simple enough to get a bound on the rate of divergence and then use our reduction cycle to lower the order until convergence. The three lemmas below correspond to the three stages in the reduction cycle. As usual, C denotes a generic constant independent of both j and k .

Lemma 1. For $\alpha > 0$, $\sigma \geq 0$, and $r \in \mathbf{R}$ the bound

$$\left| f_{j,k}^{[p]} \right| \leq C j^\sigma \frac{\alpha^j}{(d_{j,k})^r}$$

is equivalent to the bound

$$\left| g_{j,k}^{[p]} \right| \leq C' j^\sigma \frac{\alpha^j}{(d_{j,k})^{r-1}}.$$

Proof. The result follows directly from the definitions and the assumption that $\gamma < \infty$. □

The form of these bounds may look odd at first sight; the extra factor j^σ did not occur in (17) or (18). However, factors of this type can arise naturally as we go through our three-step

cycle. For the cubic Lagrange interpolation case, for instance, we shall see below that after one three-stage cycle (18) leads to

$$\left| f_{j,k}^{[3]} \right| \leq C j \frac{(1-\beta)^j}{d_{j,k}^2}.$$

In order to use Lemma 1 in the next cycle for this example, we therefore need to be able to handle such extra factors j^σ .

Lemma 2. *Suppose that, for some $\alpha > 0$, $\sigma \geq 0$, and $r \in \mathbf{R}$,*

$$\left| g_{j,k}^{[p+1]} \right| \leq C j^\sigma \frac{\alpha^j}{(d_{j,k})^r}.$$

Also, let the coefficients of the subdivision scheme $S^{[p]}$ be bounded uniformly in j, k, l . Then

$$\left| f_{j+1,2k+s}^{[p]} - f_{j,k}^{[p]} \right| \leq C' j^\sigma \frac{\alpha^j}{(d_{j,k})^r}$$

for $s \in \{0, 1\}$.

Proof.

$$\left| f_{j+1,2k+s}^{[p]} - f_{j,k}^{[p]} \right| = \left| \sum_{t=\lceil \frac{s}{2} - \frac{B}{2} \rceil}^{\lfloor \frac{s}{2} + \frac{B}{2} \rfloor} S_{j,2k+s,k+t}^{[p]} (f_{j,k+t}^{[p]} - f_{j,k}^{[p]}) \right| \leq C' j^\sigma \frac{\alpha^j}{(d_{j,k})^r},$$

where we have used that

- the sum is finite,
- the coefficients of the subdivision matrix are uniformly bounded,
- $\sum_m S_{j,l,m}^{[p]} = 1$,
- because of homogeneity the $d_{j,k+t}$, are bounded below by $C d_{j,k}$, with $C > 0$.

□

Lemma 3. *Suppose that, for some $\alpha > 0$, $\sigma \geq 0$, and $r \geq 0$,*

$$\left| f_{j+1,2k+s}^{[p]} - f_{j,k}^{[p]} \right| \leq C j^\sigma \frac{\alpha^j}{(d_{j,k})^r}$$

for all $s \in \{0, 1\}$. Then

$$\left| f_{j,k}^{[p]} \right| \leq C' \left[j^{\sigma+\omega} \frac{\hat{\alpha}^j}{(d_{j,k})^r} + 1 \right], \tag{20}$$

where $\hat{\alpha} = \max \{ \alpha, (1-\beta)^r \}$ and

$$\omega = \begin{cases} 0 & \text{if } \alpha \neq (1-\beta)^r, \\ 1 & \text{if } \alpha = (1-\beta)^r. \end{cases}$$

Proof. Write $k = 2^j b_0 + 2^{j-1} b_1 + \dots + 2b_{j-1} + b_j$, where $b_n \in \{0, 1\}$ for $n \geq 1$. Then

$$\begin{aligned} \left| f_{j, 2^j b_0 + 2^{j-1} b_1 + \dots + 2b_{j-1} + b_j}^{[p]} \right| &\leq \left| f_{0, b_0}^{[p]} \right| + \sum_{q=0}^{j-1} \left| f_{q+1, 2^{q+1} b_0 + \dots + b_{q+1}}^{[p]} - f_{q, 2^q b_0 + \dots + b_q}^{[p]} \right| \leq \\ &\leq C + \sum_{q=0}^{j-1} C q^\sigma \frac{\alpha^q}{(d_{q, 2^q b_0 + \dots + b_q})^r} \\ &\leq C + C j^\sigma \frac{1}{(d_{j-1, [k/2]})^r} \sum_{q=0}^{j-1} \alpha^q \left(\frac{d_{j-1, 2^{j-1} b_0 + \dots + b_{j-1}}}{d_{q, 2^q b_0 + \dots + b_q}} \right)^r \end{aligned}$$

Now, $d_{n+1, 2^{n+1} b_0 + \dots + b_{n+1}} \leq (1 - \beta) d_{n, 2^n b_0 + \dots + b_n}$, and thus we have $d_{j-1, 2^{j-1} b_0 + \dots + b_{j-1}} \leq (1 - \beta)^{j-1-q} d_{q, 2^q b_0 + \dots + b_q}$. Also $d_{j-1, [k/2]}$ is commensurate with $d_{j,k}$ so that

$$\left| f_{j,k}^{[p]} \right| \leq C + C j^\sigma \frac{1}{d_{j,k}^r} \sum_{q=0}^{j-1} \alpha^q ((1 - \beta)^r)^{j-1-q}$$

and the result then follows easily. \square

The cycle given by Lemmas 1-3 now provides the mechanism to translate the growth rate of the $f_{j,k}^{[p]}$ differences into a reduced growth rate for lower order divided differences $f_{j,k}^{[p-1]}$. Note that the extra constant introduced by Lemma 3 can be ignored given that all bounds diverge. Typically the exponent r gets reduced by one every time through the cycle. Note that there are two ways in which a bound of the form $\alpha^j d_{j,k}^{-r}$ can go to zero: if $r \geq 0$, and $\alpha < \beta^r$, or also if $r < 0$, $\alpha < (1 - \beta)^r$; this second case allows $\alpha > 1$. We will keep going through this cycle till we have reduced r enough so that we are one cycle short of convergence, i.e until we have either $r \geq 1$, $\alpha < \beta^{r-1}$ or $r < 1$, $\alpha < (1 - \beta)^{r-1}$. At that point we shall invoke Theorem 4 which will handle the transition to convergence and compute the fractional regularity.

There is one more issue that we need to address before stating the convergence theorem and it involves some fine tuning of the parameters for Lemma 3. Consider the case where $r > 0$, and α is small in the sense that $\alpha < (1 - \beta)^r$. This small α will get wiped out by Lemma 3 as the result will be of the form $(1 - \beta)^{j^r} / d_{j,k}^r$. Thus we have “wasted” a good, i.e., small α . However, it would be nice if we could tune our parameters in the original bound by increasing α in the denominator and balancing this off by reducing r in the numerator, till the new α becomes equal to the new $(1 - \beta)^r$. The result is then again of the form $(1 - \beta)^{j^r} / d_{j,k}^r$, but now with the new, smaller r . Thus our good α did not get wasted, but contributed to reducing r further. This can be done by finding $\tau > 0$, so that $\alpha = \beta^\tau (1 - \beta)^{r-\tau}$. Then

$$\frac{\alpha^j}{d_{j,k}^r} = \frac{\beta^{\tau j} (1 - \beta)^{(r-\tau)j}}{d_{j,k}^r} \leq C' \frac{(1 - \beta)^{(r-\tau)j}}{d_{j,k}^{r-\tau}}.$$

Thus τ is exactly the amount by which r gets reduced. If $\alpha < (1 - \beta)^r$ a $\tau > 0$ exists and is given by

$$\tau = \frac{\log \alpha - r \log(1 - \beta)}{\log \beta - \log(1 - \beta)}.$$

Note that if the third stage results in a bound of the form $(1 - \beta)^{j^r} / d_{j,k}^r$, then, in the next cycle, the bound of the second stage will be of the form $(1 - \beta)^{j^r} / d_{j,k}^{r-1}$ which means that in the next application of Lemma 3 α is strictly less than $(1 - \beta)^{r-1}$ and we need fine tuning again.

As mentioned earlier we can go through the cycle a number of times till $r \geq 1$ and $\alpha < \beta^{r-1}$ or $r < 1$ and $\alpha < (1 - \beta)^{r-1}$. Then we can use the following theorem.

Theorem 4. *Suppose that, for some $P \geq 1$, $\sigma \geq 0$, $r \in \mathbf{R}$,*

$$\left| f_{j,k}^{[P]} \right| \leq C j^\sigma \frac{\alpha^j}{(d_{j,k})^r}, \quad (21)$$

and that either $r \geq 1$, $\alpha < \beta^{r-1}$ or $r < 1$, $\alpha < (1 - \beta)^{r-1}$. Then the limit functions $\varphi^{[p]}(y)$ as defined by (13) are well-defined and continuous for all $p = 0, \dots, P - 1$. Moreover, $\varphi^{[P-1]}$ is Hölder-continuous with Hölder exponent $1 - r + \log \alpha / \log \beta - \sigma \epsilon > 0$. The rate of convergence is exponential:

$$\left| \varphi^{[P-1]}(y) - f_{j,k_j(y)}^{[P-1]} \right| \leq C' (\alpha \zeta)^j,$$

where

$$\zeta = \begin{cases} (1 - \beta)^{1-r} & \text{if } r < 1, \\ \beta^{1-r} & \text{if } r \geq 1. \end{cases}$$

Finally, one has

$$\varphi^{[p]}(y) = \frac{1}{p!} \frac{d^p \varphi^{[0]}(y)}{dy^p} \quad \text{for } p = 0, \dots, P - 1. \quad (22)$$

Proof. Applying Lemmas 1-2 once more to (21) leads to

$$\left| f_{j+1,k_{j+1}(y)}^{[P-1]} - f_{j,k_j(y)}^{[P-1]} \right| \leq C j^\sigma \alpha^j d_{j,k_j(y)}^{1-r}, \quad (23)$$

where we remember from Section 3 that $k_j(y) = \max\{l : x_{j,l} \leq x\}$. Since $k_{j+1}(y)$ is either $2k_j(y)$ or $2k_j(y) + 1$, we can write $\beta d_{q,k_q(z)} \leq d_{q+1,k_{q+1}(z)} \leq (1 - \beta) d_{q,k_q(z)}$. Therefore, $d_{q+1,k_{q+1}(z)}^{1-r} \leq \zeta d_{q,k_q(z)}^{1-r}$ is always true, with ζ defined as above. Hence from (23) we have

$$\left| f_{j+1,k_{j+1}(y)}^{[P-1]} - f_{j,k_j(y)}^{[P-1]} \right| \leq C' j^\sigma (\alpha \zeta)^j. \quad (24)$$

Given that $0 < \alpha \zeta < 1$, it follows that $(f_{j,k_j(y)}^{[P-1]})_{j \in \mathbf{N}}$ is a Cauchy sequence, and the function $\varphi^{[P-1]}(y)$ is well-defined.

Next we estimate $|\varphi^{[P-1]}(y+t) - \varphi^{[P-1]}(y)|$ for t satisfying $|t| < \min_k d_{0,k}$. Define $h_j(y) = \min(d_{j,k_j(y)-1}, d_{j,k_j(y)}, d_{j,k_j(y)+1})$. The sequence $h_j(y)$ need not be monotone decreasing, but its lim sup equals zero. We can therefore find j so that

$$h_j(y) > |t| \geq h_{j+1}(y).$$

It then follows that $|k_j(y+t) - k_j(y)| \leq 1$, so that

$$\left| f_{j,k_j(y+t)}^{[P-1]} - f_{j,k_j(y)}^{[P-1]} \right| \leq C j^\sigma \alpha^j d_{j,k_j(y)}^{1-r}$$

by Lemma 1. By homogeneity, we conclude that

$$C_1 |t| \leq d_{j,k_j(y)+s} \leq C_2 |t| \quad \text{for } s \in \{-1, 0, 1\}. \quad (25)$$

Moreover, for any z ,

$$\begin{aligned} \left| \varphi^{[P-1]}(z) - f_{j,k_j(z)}^{[P-1]} \right| &\leq \sum_{q=j}^{\infty} \left| f_{q+1,k_{q+1}(z)}^{[P-1]} - f_{q,k_q(z)}^{[P-1]} \right| \leq C \sum_{q=j}^{\infty} q^\sigma \alpha^q (d_{q,k_q(z)})^{1-r} \\ &\leq C' \alpha^j (d_{j,k_j(z)})^{1-r}. \end{aligned} \quad (26)$$

Consequently

$$\begin{aligned} \left| \varphi^{[P-1]}(y+t) - \varphi^{[P-1]}(y) \right| &\leq \left| \varphi^{[P-1]}(y+t) - f_{j,k_j(y+t)}^{[P-1]} \right| + \left| f_{j,k_j(y+t)}^{[P-1]} - f_{j,k_j(y)}^{[P-1]} \right| + \\ &\quad \left| f_{j,k_j(y)}^{[P-1]} - \varphi^{[P-1]}(y) \right| \leq C'' j^\sigma \alpha^j \left(d_{j,k_j(y+t)}^{1-r} + d_{j,k_j(y)}^{1-r} \right) \leq C''' j^\sigma \alpha^j |t|^{1-r}, \end{aligned}$$

by (25). On the other hand, $h_{j+1}(y) \geq C\beta^{j+1}$, hence $|t| \geq C'\beta^j$, or $\alpha^j \leq C|t|^{\log \alpha / \log \beta}$. Altogether, this implies

$$\left| \varphi^{[P-1]}(y+t) - \varphi^{[P-1]}(y) \right| \leq C''' (1 + |\log |t||)^\sigma |t|^{1-r+\log \alpha / \log \beta}.$$

Also, (26) implies an exponential rate of convergence of $f_{j,k}^{[P-1]}$ to its limit function, namely

$$\left| \varphi^{[P-1]}(y) - f_{j,k_j(y)}^{[P-1]} \right| \leq C' \alpha^j (d_{j,k_j(z)})^{1-r} \leq C' (\alpha \zeta)^j.$$

Now we can show the convergence of $f_{j,k}^{[p]}$ for $p = 0, \dots, P-2$. It follows from (24) that the sequence $f_{j,k}^{[P-1]}$ is uniformly bounded, i.e., $f_{j,k}^{[P-1]} \leq C'$. Here we can apply the first part of the theorem again to show that $f_{j,k}^{[P-2]}$ converges to a continuous function. We can also repeat this argument for all the lower p . Lemma 5 below provides the proof for (22). \square

Lemma 5. *Suppose that the divided differences $f_{j,k}^{[p]}$ converge uniformly to the continuous functions $\varphi^{[p]}(y)$ for $p = 0, \dots, P-1$. Then*

$$\varphi^{[p]}(y) = \frac{1}{p!} \frac{d^p \varphi^{[0]}(y)}{dy^p}$$

for $p = 1, \dots, P-1$.

This lemma concerns divided differences rather than subdivision. For completeness we give a proof in Appendix C.

7. ESTIMATES FOR HIGHER-ORDER DIFFERENCES IN THE CASE OF CUBIC LAGRANGE INTERPOLATION

Previous section introduced a general reduction procedure that takes a bound on the growth of higher-order divided differences and transforms it into the regularity estimates for the limit function of the subdivision. Here we shall prove several bounds on the growth of the fourth divided differences in the case of the cubic Lagrange interpolation scheme of Section 5. Results in this section (except where specified otherwise) are generally true for any dyadically balanced

multi-level grid. This manifest itself, for instance, in the presence of β and the absence of γ in the statements of Lemmas 6 and 7.

7.1. Notation. Whereas the forward divided differences were sufficient for the algebraic manipulations in the preceding sections, the more meticulous estimates necessary for the inhomogeneous case are more natural with a centered notation. We thus introduce

$$\begin{aligned} G_{j,k}^{[p]} &\equiv g_{j,k-\lfloor \frac{p}{2} \rfloor}^{[p]}, \\ F_{j,k}^{[p]} &\equiv f_{j,k-\lfloor \frac{p}{2} \rfloor}^{[p]}, \\ D_{j,k}^{[p]} &\equiv d_{j,k-\lfloor \frac{p}{2} \rfloor}^{[p]}. \end{aligned}$$

Obviously, for $p = 0$ or 1 the notations coincide.

7.2. Bounds for $g_{j,k-2}^{[4]}$. It will be convenient, in the sequel, to have a shorthand notation for distances at level $j + 1$ near point $x_{j,k} = x_{j+1,2k}$. We shall denote by $[s, t]$ the distance

$$[s, t] = x_{j+1,2k+t} - x_{j+1,2k+s} = d_{j+1,2k+s}^{[t-s]}.$$

Let's illustrate this notation by rewriting the subdivision scheme $T^{[4]}$ for the differences of the third divided differences for the cubic Lagrange interpolation as computed in Section 5:

$$G_{j+1,2k}^{[4]} = \frac{[-2, 2]}{[-1, 1]} G_{j,k}^{[4]}, \quad (27)$$

$$G_{j+1,2k+1}^{[4]} = -\frac{[-2, -1]}{[-1, 1]} G_{j,k}^{[4]} - \frac{[3, 4]}{[1, 3]} G_{j,k+1}^{[4]}. \quad (28)$$

This scheme can be represented by Figure 5. The following lemma gives a bound on the growth

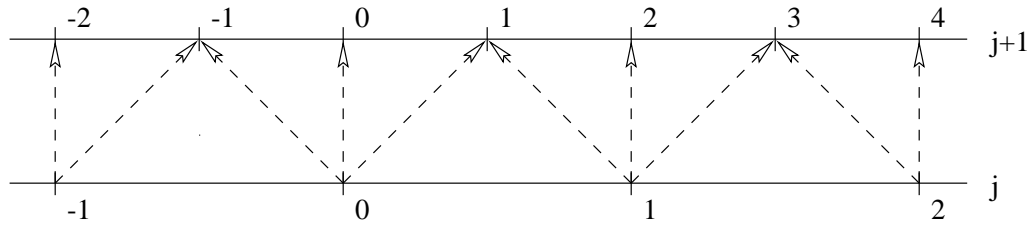


FIGURE 5. Subdivision scheme for $G_{[4]}$.

of $|G_{j,k}^{[4]}|$ as j increases.

Lemma 6. *Suppose that $\beta > 0$. Then*

$$|G_{j,k}^{[4]}| \leq C \frac{(1 - \beta)^j}{d_{j,k-1} d_{j,k}}, \quad (29)$$

where C is a constant independent of j and k .

Proof. First of all, since initial data are finitely supported, the estimate trivially holds on the level $j = 0$ with some constant $C > 0$.

Suppose that on the j -th level the estimate holds for all k , i.e.:

$$\left| G_{j,k}^{[4]} \right| \leq C \frac{(1-\beta)^j}{[-2,0][0,2]} \quad \text{and} \quad \left| G_{j,k+1}^{[4]} \right| \leq C \frac{(1-\beta)^j}{[0,2][2,4]}.$$

First, consider the odd point: from (28) it follows that

$$\begin{aligned} \left| G_{j+1,2k+1}^{[4]} \right| &\leq \left(\frac{[-2,-1]}{[-1,1]} \frac{1}{[-2,0]} + \frac{[3,4]}{[1,3]} \frac{1}{[2,4]} \right) \frac{C(1-\beta)^j}{[0,2]} \\ &= \left([1,2] \frac{[-2,-1]}{[-2,0]} \frac{[0,1]}{[-1,1]} + [0,1] \frac{[3,4][1,2]}{[2,4][1,3]} \right) \frac{1}{[0,2]} \frac{C(1-\beta)^j}{[0,1][1,2]}. \end{aligned} \quad (30)$$

The definition of β implies that

$$\frac{[-2,-1]}{[-2,0]} \leq 1-\beta, \quad \frac{[0,1]}{[-1,1]} \leq 1, \quad \frac{[3,4]}{[2,4]} \leq 1-\beta, \quad \frac{[1,2]}{[1,3]} \leq 1. \quad (31)$$

Substituting (31) into (30) yields

$$\left| G_{j+1,2k+1}^{[4]} \right| \leq (1-\beta) ([1,2] + [0,1]) \frac{1}{[0,2]} \frac{C(1-\beta)^j}{[0,1][1,2]} = (1-\beta) \frac{C(1-\beta)^j}{[0,1][1,2]} = \frac{C(1-\beta)^{j+1}}{d_{j+1,2k} d_{j+1,2k+1}}.$$

Consider the even point now: from (27) it follows that

$$\left| G_{j+1,2k}^{[4]} \right| \leq \frac{[-2,2]}{[-1,1]} \frac{C(1-\beta)^j}{[-2,0][0,2]} = \frac{[-2,2]}{[-1,1]} \frac{[-1,0]}{[-2,0]} \frac{[0,1]}{[0,2]} \frac{C(1-\beta)^j}{[-1,0][0,1]}.$$

$$\text{Now either } \frac{[-1,0]}{[-2,0]} \leq \frac{[-1,1]}{[-2,2]}, \quad \text{or} \quad \frac{[0,1]}{[0,2]} \leq \frac{[-1,1]}{[-2,2]},$$

and therefore,

$$\left| G_{j+1,2k}^{[4]} \right| \leq (1-\beta) \frac{C(1-\beta)^j}{[-1,0][0,1]} = \frac{C(1-\beta)^{j+1}}{d_{j+1,2k-1} d_{j+1,2k}}.$$

The lemma follows by induction. \square

This lemma gives the starting estimate needed for the reduction process and the regularity estimates of the previous section. Note that the proof did not use homogeneity at all. Lemma 6 holds even if $\gamma = \infty$, as long as $\beta > 0$. We also use the following estimate which is stronger than Lemma 6, and is necessary for the analysis of the inhomogeneous case.

Lemma 7. *Suppose that $\beta > 0$. Then*

$$\left| G_{j,k}^{[4]} \right| \leq C \frac{(1-\beta)^j}{\left(D_{j,k}^{[2]} \right)^2},$$

where C is a constant independent of j and k .

Proof. First of all, the lemma trivially holds on the level $j = 0$. Then, consider the case when $j > 0$ and k is odd. From the previous lemma we have

$$\left| G_{j,k}^{[4]} \right| \leq C \frac{(1-\beta)^j}{d_{j,k-1} d_{j,k}}$$

Since k is odd, we know that the intervals $(x_{j,k-1}, x_{j,k})$ and $(x_{j,k}, x_{j,k+1})$ are the two ‘‘children’’ of the same interval from the previous level, hence $d_{j,k-1} \geq \beta d_{j-1, \lfloor k/2 \rfloor}$ and $d_{j,k} \geq \beta d_{j-1, \lfloor k/2 \rfloor}$. But $d_{j-1, \lfloor k/2 \rfloor} \equiv D_{j,k}^{[2]}$, therefore

$$\left| G_{j,k}^{[4]} \right| \leq C' \frac{(1-\beta)^j}{\left(D_{j,k}^{[2]} \right)^2}, \text{ where } C' = C/\beta^2.$$

Now, consider the case $j > 0$ and k is even. Suppose that on the level j the result holds. Consider an even point on the $(j+1)$ -th level. From (27) we have

$$\left| G_{j+1,2k}^{[4]} \right| = \frac{D_{j,k}^{[2]}}{D_{j+1,2k}^{[2]}} \left| G_{j,k}^{[4]} \right| \leq \frac{D_{j,k}^{[2]}}{D_{j+1,2k}^{[2]}} C \frac{(1-\beta)^j}{\left(D_{j,k}^{[2]} \right)^2} = C \frac{D_{j+1,2k}^{[2]}}{D_{j,k}^{[2]}} \frac{(1-\beta)^j}{\left(D_{j+1,2k}^{[2]} \right)^2}.$$

Also, $D_{j+1,2k}^{[2]} = d_{j+1,2k-1} + d_{j+1,2k} \leq (1-\beta)d_{j,k-1} + (1-\beta)d_{j,k} = (1-\beta)D_{j,k}^{[2]}$.

Therefore, the estimate is established on the $(j+1)$ -th level at all the even points. The result follows by induction. \square

Under extra restrictions on γ or β , these bounds can be improved further.

Lemma 8. *Suppose that $\beta \geq 1/3$ (non homogeneous but dyadically balanced case) or that $\gamma \leq \gamma_0 \approx 2.4992$ (homogeneous case). Then*

$$\left| G_{j,k}^{[4]} \right| \leq \frac{C}{D_{j,k}^{[2]}},$$

where C is a constant independent of j and k .

Proof. We work again by induction on j . Suppose that on the j -th level the estimate holds for all k , i.e.

$$\left| G_{j,k}^{[4]} \right| \leq \frac{C}{[-2, 2]} \quad \text{and} \quad \left| G_{j,k+1}^{[4]} \right| \leq \frac{C}{[0, 4]}.$$

Let us first check the even points at level $j+1$:

$$\left| G_{j+1,2k}^{[4]} \right| = \frac{[-2, 2]}{[-1, 1]} \left| G_{j,k}^{[4]} \right| \leq \frac{C}{[-1, 1]} = \frac{C}{D_{j+1,2k}^{[2]}}.$$

Now for the odd point:

$$\left| G_{j+1,2k+1}^{[4]} \right| = \left| -\frac{[-2, -1]}{[-1, 1]} G_{j,k}^{[4]} - \frac{[3, 4]}{[1, 3]} G_{j,k+1}^{[4]} \right| \leq \frac{C}{[0, 2]} \left\{ \frac{[-2, -1]}{[-1, 1]} \frac{[0, 2]}{[-2, 2]} + \frac{[3, 4]}{[1, 3]} \frac{[0, 2]}{[0, 4]} \right\} \quad (32)$$

This is less than $C/[0, 2] = C/D_{j+1, 2k+1}^{[2]}$, proving the induction step, if the expression between curly brackets is bounded by 1. Let us check this remaining factor. We first concentrate on the case $\beta \geq 1/3$. Because, for $a, b \geq 0$, $ab \leq (a+b)^2/4$, we have

$$\frac{[3, 4][0, 2]}{[1, 3][0, 4]} \leq \frac{1 [3, 4][0, 4]}{4 [2, 4][1, 3]}.$$

Since $[x_{j+1, 2k+3}, x_{j+1, 2k+4}]$ is a child interval of $[x_{j+1, 2k+2}, x_{j+1, 2k+4}] = [x_{j, k+1}, x_{j, k+2}]$, we have $[3, 4] \leq (1 - \beta) [2, 4]$.

Similarly $[1, 2] \geq \beta [0, 2]$, $[2, 3] \geq \beta [2, 4]$. Consequently $[1, 3] = [1, 2] + [2, 3] \geq \beta [0, 4]$, and we have

$$\frac{[3, 4][0, 4]}{[2, 4][1, 3]} \leq \frac{1 - \beta}{\beta}.$$

Similar bounds hold for the other term in the curly brackets in (32), so that

$$\left\{ \frac{[-2, -1] [0, 2]}{[-1, 1] [-2, 2]} + \frac{[3, 4] [0, 2]}{[1, 3] [0, 4]} \right\} \leq \frac{1 - \beta}{2\beta}.$$

This does not exceed 1 if $\beta \geq 1/3$, which proves the induction step in this case.

Since $\beta \geq (1 + \gamma)^{-1}$, the previous argument already proves the lemma for $\gamma \leq 2$ in the homogeneous case as well. In Appendix D we show that in the homogeneous case one can extend the range for γ slightly, to $\gamma \leq \gamma_0 \approx 2.4992$. \square

8. REGULARITY OF THE CUBIC LAGRANGE SUBDIVISION IN THE HOMOGENEOUS CASE

In this section, we apply the machinery of Section 6 to cubic Lagrange interpolation subdivision scheme. This is a scheme of order 4, and we have a simple matrix structure for $S^{[4]}$, in which each column has three elements. The expression for these elements is even simpler if, at the last stage, we consider simply differences and not divided differences, corresponding to (14), with the operator $T^{[4]}$ as defined by (15) and (16). In section 7, it was proved that (see Lemma 6; beware of the change in notation: $g_{j, k}^{[4]} = G_{j, k+2}^{[4]}$)

$$\left| g_{j, k}^{[4]} \right| \leq C \frac{(1 - \beta)^j}{d_{j, k+1} d_{j, k+2}}.$$

Because of homogeneity, this can be rewritten as

$$\left| g_{j, k}^{[4]} \right| \leq C' \frac{(1 - \beta)^j}{(d_{j, k})^2},$$

or by Lemma 1

$$\left| f_{j, k}^{[4]} \right| \leq C' \frac{(1 - \beta)^j}{(d_{j, k})^3}.$$

We can now apply the machinery proved earlier. In this example the initial bound for Lemma 1 has $p = 4, r = 3, \alpha = 1 - \beta, \sigma = 0$; after two reduction cycles we arrive at $P = p = 2, r = 1, \alpha = 1 - \beta, \sigma = 1$ for use in Theorem 4, i.e., we have the estimate

$$\left| f_{j, k}^{[2]} \right| \leq C j \frac{(1 - \beta)^j}{d_{j, k}}.$$

From the theorem we conclude that the subdivision scheme $S^{[0]}$ converges to a C^1 limit function $\varphi = \varphi^{[0]}$, and that φ' is Hölder continuous with Hölder exponent $\log(1 - \beta)/\log \beta - \epsilon$, for $\epsilon > 0$. In fact, we even have

$$|\varphi'(x + t) - \varphi'(x)| \leq C (1 + |\log |t||) |t|^{\log(1-\beta)/\log \beta} \quad (33)$$

In the equally spaced case and in the semi-regular case we have $\beta = 1/2$, and we recover the optimal estimate from [16].

If $\gamma \leq \gamma_0 \approx 2.4492$, then we can do better by using Lemma 8 in Section 7. In this case, we have the stronger estimate

$$|g_{j,k}^{[4]}| \leq \frac{C}{d_{j,k+1}^{[2]}},$$

which can again, because of homogeneity and Lemma 1 be rewritten as

$$|f_{j,k}^{[4]}| \leq \frac{C'}{(d_{j,k})^2}.$$

We can then apply our cycle twice and conclude

$$|f_{j,k}^{[2]}| \leq Cj, \quad (34)$$

corresponding to $P = 2$, $r = 0$, $\alpha = 1$, $\sigma = 1$ in Theorem 4. Hence, we find that the $f_{j,k}^{[1]}$ converge to a continuous function $\varphi^{[1]} = \varphi'$ with Hölder exponent $1 - \epsilon$, with ϵ arbitrarily small; in fact $|\varphi'(x + t) - \varphi'(x)| \leq C (1 + |\log |t||) |t|$.

So for $\gamma \leq \gamma_0$, cubic Lagrange interpolation leads to a limit function φ that is $C^{2-\epsilon}$, which is the same regularity as in the regularly spaced case. For $\gamma > \gamma_0$, our methods prove the weaker regularity result (33), but we conjecture that this estimate is not sharp. Note that although we used homogeneity repeatedly in the proofs of the lemmas and the theorem, the final Hölder exponent depends on β only, not on γ . Figure 6 summarizes the bounds proved in this paper on the Hölder regularity of the limit function as a function of β . Even though there is no mathematical evidence we think that it is rather unlikely for the Hölder exponent to be discontinuous; for $\beta < 1/3$, we thus do not believe this result to be optimal. We conjecture that φ remains $C^{2-\epsilon}$ for all $\beta > 0$.

9. REGULARITY OF THE CUBIC LAGRANGE SUBDIVISION WITHOUT HOMOGENEITY

In this section we shall re-examine the very specific case of cubic Lagrange interpolation without assuming homogeneity. It will turn out that we can derive the same conclusion as before, with the same Hölder exponent, even if the grid is only dyadically balanced and not homogeneous. We follow the same game plan as before, except that we have to be much more careful: the absence of homogeneity requires a detailed and careful analysis at each of the reduction steps.

9.1. Reduction procedure. Without homogeneity the reduction procedure becomes much more delicate. We can no longer use a general set of lemmas, but instead have to consider every single step in the reduction cycle for the particular case of cubic Lagrange subdivision. We start by noting that a constant C_S , which only depends on β , exists so that the coefficients of the

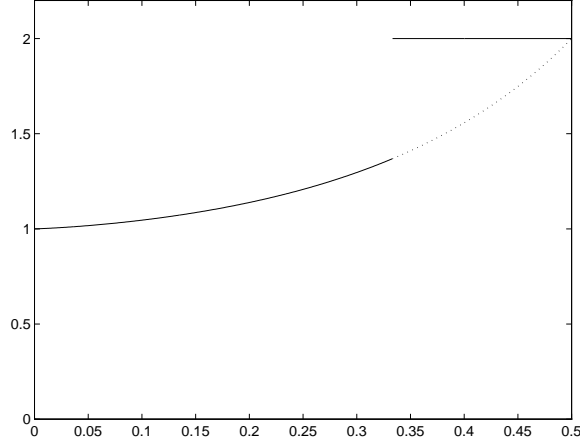


FIGURE 6. Hölder regularity in function of β .

derived subdivision schemes $S_j^{[p]}$ for $p = 1, 2, 3$ are uniformly bounded by C_S . To get to our final estimate on $F^{[1]}$ we will need 8 steps.

Step 1: Starting point

Earlier we showed the following estimate for $G_{[4]}$.

$$\left| G_{j,k}^{[4]} \right| \leq C \frac{(1-\beta)^j}{\left(D_{j,k}^{[2]} \right)^2}$$

Step 2: From $\left| G_{j,k}^{[4]} \right|$ to $\left| F_{j+1,2k+t}^{[3]} - F_{j,k}^{[3]} \right|$

Because $S^{[3]}$ is bounded uniformly in j , and conserves constants, one can show for all j and k :

$$\left| F_{j+1,2k+t}^{[3]} - F_{j,k+s}^{[3]} \right| \leq C_S \left| G_{j,k}^{[4]} \right| \leq C_S C \frac{(1-\beta)^j}{\left(D_{j,k}^{[2]} \right)^2} \quad \text{for } s = -1, 0 \text{ and } t = -1, 0 \quad (35)$$

We have to restrict the possible values of s and t because we are not assuming homogeneity. Considering a wider range of s and t would bring other $D_{j,l}^{[2]}, l \neq k$ into play, which could not be related to $D_{j,k}^{[2]}$ in a way independent of j and k .

Step 3: Summing backwards for $F^{[3]}$

In the proof of Lemma 3, we used an estimate of type (35) to derive a bound on $F_{j,k}^{[3]}$ by summing “backwards”, that is, over q , with $0 \leq q \leq j$. When the mesh was homogeneous, and we could let the indices s and t in (35) range over a larger set, this was easy. Now that we no longer assume homogeneity, we have to make sure that (35) for $s, t \in \{-1, 0\}$ is sufficient to allow us to carry out the backwards summation, ranging all the way to the coarsest level, starting from arbitrary k . We describe below how to choose a “path”, from level $j+1$ backwards to level 0,

that allows backwards summation. The main idea is to use the freedom of choosing s in (35) to ensure that $D_{j-1, \lceil (k+s)/2 \rceil}^{[2]} / D_{j,k}^{[2]}$ is bounded by $1 - \beta$.

Let us start with arbitrary j, k . Then $k = k_{j+1} = 2l_j + s_j$, where $l_j = \lceil k_{j+1}/2 \rceil$, $s_j \in \{0, -1\}$. It then follows that

$$\left| F_{j+1, k_{j+1}}^{[3]} - F_{j, l_j + t_j}^{[3]} \right| \leq \frac{C(1 - \beta)^j}{\left(D_{j, l_j}^{[2]} \right)^2}, \quad (36)$$

where we can still choose $t_j \in \{-1, 0\}$ freely. Whatever choice we make, we will have $k_j = l_j + t_j = 2l_{j-1} + s_{j-1}$, with $l_{j-1} = \lceil (l_j + t_j)/2 \rceil$, $s_{j-1} \in \{-1, 0\}$, and therefore

$$\left| F_{j, l_j + t_j}^{[3]} - F_{j-1, l_{j-1} + t_{j-1}}^{[3]} \right| \leq \frac{C(1 - \beta)^{j-1}}{\left(D_{j-1, l_{j-1}}^{[2]} \right)^2}, \quad (37)$$

where again t_{j-1} can be chosen freely in $\{-1, 0\}$. In order to make the argument in Lemma 3 work, we will need to bound $D_{j, l_j}^{[2]}$ by $(1 - \beta)D_{j-1, l_{j-1}}^{[2]}$, which is only possible if there is some hierarchical nesting. This will fix the choice for t_j . In particular, if l_j is even, $l_j = 2m_j$, then $D_{j, l_j}^{[2]} \leq (1 - \beta)D_{j-1, m_j}^{[2]}$, so we want $l_{j-1} = m_j = l_j/2$. This corresponds to the choice $t_j = 0$. If $l_j = 2m_j + 1$ is odd, then $D_{j, l_j}^{[2]} = d_{j-1, m_j}^{[1]} \leq (1 - \beta)d_{j-2, \lfloor m_j/2 \rfloor}^{[1]} = (1 - \beta)D_{j-1, 2\lfloor m_j/2 \rfloor + 1}^{[2]}$. In this case we therefore want $l_{j-1} = 2\lfloor m_j/2 \rfloor + 1$. This implies that $l_j + t_j = 2l_{j-1} + s_{j-1} = 4\lfloor m_j/2 \rfloor + 2 + s_{j-1}$ has to be equal to 1 or 2 (mod 4). This can always be achieved by an appropriate choice of t_1 : if $l_j = 1 \pmod{4}$, we choose $t_j = 0$, if $l_j = 3 \pmod{4}$, we choose $t_j = -1$.

We can summarize the procedure as follows:

$$k_j = \left\lfloor \frac{k_{j+1}}{2} \right\rfloor + t_j,$$

$$t_j = \begin{cases} 0 & \text{if } \lceil k_{j+1}/2 \rceil \text{ is even,} \\ 0 & \text{if } \lceil k_{j+1}/2 \rceil = 1 \pmod{4}, \\ -1 & \text{if } \lceil k_{j+1}/2 \rceil = 3 \pmod{4}. \end{cases}$$

In all cases we have achieved our goal of writing (36) and (37) with appropriate choices such that $D_{j, l_j}^{[2]} \leq (1 - \beta)D_{j-1, l_{j-1}}^{[2]}$. At this stage, t_{j-1} is still free, but it will likewise be fixed by the next stage, where we will want a similar inequality again, now involving a $D_{j-2, l_{j-2}}^{[2]}$ that should be at least as large as $(1 - \beta)^{-1}D_{j-1, l_{j-1}}^{[2]}$. We continue in this vein until we reach the coarsest level. This defines an unambiguous procedure to find an appropriate backwards path. Figure 7 illustrates the first three steps of these backwards paths.

Along this backwards path, we have, at every step,

$$D_{q, l_q}^{[2]} \leq (1 - \beta)D_{q-1, l_{q-1}}^{[2]} \quad \text{for } q = 1, \dots, j.$$

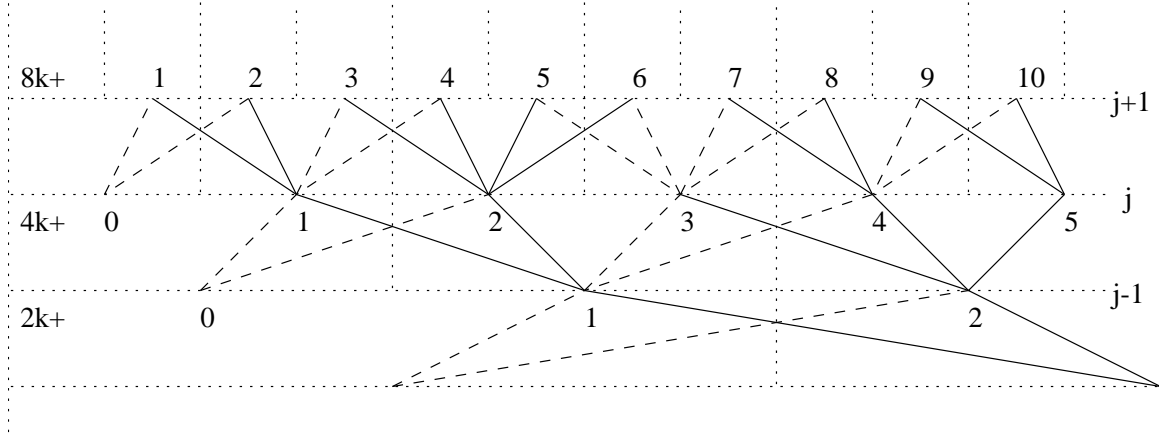


FIGURE 7. The paths followed for the choices of k_q in the backwards summation in Step 3. If pair $(j' - 1, k')$ plays a role in the subdivision that computes $F_{j',l'}^{[3]}$ at the next level, then this is indicated by a line (dashed or full) connecting $(j' - 1, k')$ and (j', l') . The backwards summation has to follow these lines, but some of the lines are “illegal”; these are dashed. By following the full lines, we guarantee that all the ratios that show up can be controlled.

This rate of decrease of “estimating” interval lengths along our path is exactly what we need for the argument in Lemma 3. As a result we obtain

$$\left| F_{j,k}^{[3]} \right| \leq C \frac{(1 - \beta)^{j-1}}{\left(D_{j-1, \lceil \frac{k}{2} \rceil}^{[2]} \right)^2}.$$

Step 4: $\left| F_{j,k}^{[3]} \right|$ to $\left| G_{j,k}^{[3]} \right|$

By definition we have:

$$G_{j,k}^{[3]} = F_{j,k}^{[3]} D_{j,k}^{[3]}$$

Moreover, considering the relative positions of the intervals, one can show that

$$D_{j,k}^{[3]} \leq D_{j-1, \lceil \frac{k}{2} \rceil}^{[2]}.$$

Hence, we get

$$\left| G_{j,k}^{[3]} \right| \leq C \frac{(1 - \beta)^{j-1}}{D_{j-1, \lceil \frac{k}{2} \rceil}^{[2]}}.$$

Step 5: $\left| G_{j,k}^{[3]} \right|$ to $\left| F_{j+1, 2k+t}^{[2]} - F_{j,k}^{[2]} \right|$

Now we have to clarify which intervals enter the estimates for $F_{j+1,l}^{[2]} - F_{j,l'}^{[2]}$. Of course, it will depend on l . First consider “odd” l . Then $l = 2u + 1$ and from the explicit expression for $S_{j;2k,l}^{[2]}$

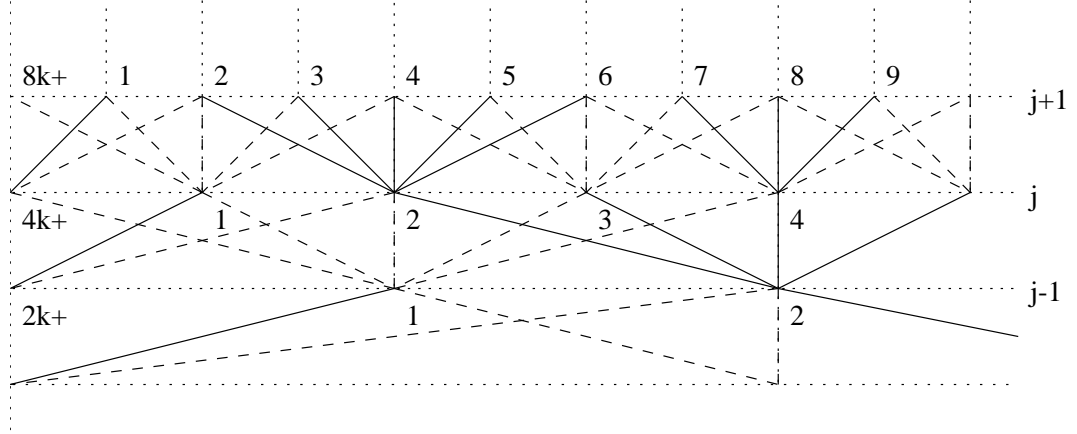


FIGURE 8. The paths followed for the choices of k_q in the backwards summation in Step 6.

in Section 5 one sees that $F_{j+1,2u+1}^{[2]}$ is a linear combination of the $F_{j,u+t}^{[2]}$, $t = 0, 1$. It follows that

$$\left| F_{j+1,2u+1}^{[2]} - F_{j,u+t}^{[2]} \right| \leq C_S \left| G_{j,u}^{[3]} \right| \leq C \frac{(1-\beta)^{j-1}}{D_{j-1, \lceil u/2 \rceil}^{[2]}} \quad \text{for } t = 0, 1. \quad (38)$$

For even l we have $l = 2u$ and $F_{j+1,2u}^{[2]}$ is a linear combination of the $F_{j,u+t}^{[2]}$, $t = -1, 0, 1$. We can subtract any of these, and estimate the difference, but now two adjacent $G_{j,l}^{[3]}$ come into play:

$$\left| F_{j+1,2u}^{[2]} - F_{j,u+t}^{[2]} \right| \leq C_S \left(\left| G_{j,u-1}^{[3]} \right| + \left| G_{j,u}^{[3]} \right| \right) \quad \text{for } t = -1, 0, 1$$

It is not immediately clear how to derive a bound of type (38) from this. We have to recognize two cases, depending on whether the bounds on $\left| G_{j,u-1}^{[3]} \right|$ and $\left| G_{j,u}^{[3]} \right|$ contain the same interval or not. The former case is easier and happens whenever $G_{j,u-1}^{[3]}$ and $G_{j,u}^{[3]}$ have two common “parents”, while the latter one requires a little bit more care and happens when $G_{j,u-1}^{[3]}$ and $G_{j,u}^{[3]}$ share only one common “parent”. To be more specific, we shall consider $l = 8k + s$ where $s = 2, 4, 6, 8$. Now, for $s = 4$ we have $u = 4k + 2$, and $\lceil u/2 \rceil = \lceil (u-1)/2 \rceil = 2k + 1$, so that

$$\left| F_{j+1,8k+4}^{[2]} - F_{j,4k+2}^{[2]} \right| \leq C \frac{(1-\beta)^{j-1}}{D_{j-1,2k+1}^{[2]}}.$$

In the same way, for $s = 8$

$$\left| F_{j+1,8k+8}^{[2]} - F_{j,4k+4}^{[2]} \right| \leq C \frac{(1-\beta)^{j-1}}{D_{j-1,2k+2}^{[2]}}.$$

Two difficult points are $s = 2$ and $s = 6$. Consider $s = 6$ with $s = 2$ following by analogy. We have

$$\left| F_{j+1,8k+6}^{[2]} - F_{j,4k+2}^{[2]} \right| \leq C(1-\beta)^{j-1} \left(\frac{1}{D_{j-1,2k+1}^{[2]}} + \frac{1}{D_{j-1,2k+2}^{[2]}} \right).$$

The following inequalities allows us to handle this case as well: we have

$$\frac{1}{D_{j-1,2k+2}^{[2]}} \leq \frac{1}{D_{j-1,2k+1}^{[1]}}$$

but also

$$D_{j-1,2k+1}^{[1]} \geq \beta D_{j-1,2k+1}^{[2]},$$

so that

$$\left| F_{j+1,8k+6}^{[2]} - F_{j,4k+2}^{[2]} \right| \leq C \left(1 + \frac{1}{\beta}\right) \frac{(1-\beta)^{j-1}}{D_{j-1,2k+1}^{[2]}}.$$

The case $s = 2$ is similar. Taking into account (38) as well, it follows that for $s = 2, 3, \dots, 6$ we can write

$$\left| F_{j+1,8k+s}^{[2]} - F_{j,4k+2}^{[2]} \right| \leq C' \frac{(1-\beta)^{j-1}}{D_{j-1,2k+1}^{[2]}},$$

whereas for $s = 7, 8, 9$ we have

$$\left| F_{j+1,8k+s}^{[2]} - F_{j,4k+4}^{[2]} \right| \leq C' \frac{(1-\beta)^{j-1}}{D_{j-1,2k+2}^{[2]}}.$$

Step 6: Summing backwards for $F^{[2]}$

Again we want to bound $\left| F_{j+1,8k+s}^{[2]} \right|$ where $s = 2, \dots, 9$. Just as we did for $F^{[3]}$ we shall obtain this bound by summing the differences $\left| F_{q+1,8l+s}^{[2]} - F_{q,4l+t}^{[2]} \right|$ “backwards”, along an appropriate path. Start again with j, k arbitrary. Then $k_{j+1} = 8l_j + s_j$, with $s_j \in \{2, 3, \dots, 9\}$, and $l_j = \lfloor (k_{j+1} - 2)/8 \rfloor$. Define

$$t(s) = \begin{cases} 2 & \text{if } s = 2, \dots, 6 \\ 4 & \text{if } s = 7, \dots, 9, \end{cases}$$

and set $k_j = 4l_j + t(s_j)$; then the result of Step 5 implies

$$\left| F_{j+1,k_{j+1}}^{[2]} - F_{j,k_j}^{[2]} \right| \leq C \frac{(1-\beta)^{j-1}}{D_{j-1,k_j/2}^{[2]}}.$$

Now k_j can be written as $k_j = 8l_{j-1} + s_{j-1}$, with $s_{j-1} \in \{2, 4, 6, 8\}$ (odd s_{j-1} are not possible since $t(s_j)$ is even). Define now $k_{j-1} = 4l_{j-1} + t(s_{j-1})$, so that

$$\left| F_{j,k_j}^{[2]} - F_{j-1,k_{j-1}}^{[2]} \right| \leq C \frac{(1-\beta)^{j-2}}{D_{j-2,k_{j-1}/2}^{[2]}}.$$

If $t(s_j) = 2$, then k_j is either $8l_{j-1} + 2$ or $8l_{j-1} + 6$, so that $k_{j-1} = 4l_{j-1} + 2$. In the first case

$$\begin{aligned} D_{j-1, k_j/2}^{[2]} &= D_{j-1, 4l_{j-1}+1}^{[2]} = d_{j-2, 2l_{j-1}} \leq (1-\beta)d_{j-3, l_{j-1}} \\ &= (1-\beta)D_{j-2, 2l_{j-1}+1}^{[2]} = (1-\beta)D_{j-2, k_{j-1}/2}^{[2]}. \end{aligned}$$

In the second case

$$D_{j-1, k_j/2}^{[2]} = D_{j-1, 4l_{j-1}+3}^{[2]} = d_{j-2, 2l_{j-1}+1} \leq (1-\beta)d_{j-3, l_{j-1}} = (1-\beta)D_{j-2, k_{j-1}/2}^{[2]}.$$

If $t(s_j) = 4$, then either $k_j = 8l_{j-1} + 4$, $k_{j-1} = 4l_{j-1} + 2$ or $k_j = 8l_{j-1} + 8$, $k_{j-1} = 4l_{j-1} + 4$. In both cases $k_j = 2k_{j-1}$, and k_{j-1} is even, so that

$$D_{j-1, k_j/2}^{[2]} = D_{j-1, k_{j-1}}^{[2]} \leq (1-\beta)D_{j-2, k_{j-1}/2}^{[2]}.$$

So in all cases $(D_{j-2, k_{j-1}/2}^{[2]})^{-1} \leq (1-\beta)(D_{j-1, k_j/2}^{[2]})^{-1}$, so that

$$\left| F_{j, k_j}^{[2]} - F_{j-1, k_{j-1}}^{[2]} \right| \leq C \frac{(1-\beta)^{j-1}}{D_{j-1, k_j/2}^{[2]}}.$$

We continue building the backwards path in this way; since at every stage we have

$$D_{q, k_{q+1}/2}^{[2]} \leq (1-\beta)D_{q-1, k_q/2}^{[2]},$$

we obtain

$$\left| F_{q+1, k_{q+1}}^{[2]} - F_{q, k_q}^{[2]} \right| \leq C \frac{(1-\beta)^{j-1}}{D_{j-1, k_j/2}^{[2]}},$$

hence

$$\left| F_{j+1, k_{j+1}}^{[2]} \right| = \left| F_{j+1, 8l_j+s_j}^{[2]} \right| \leq C \frac{j(1-\beta)^{j-1}}{D_{j-1, k_j/2}^{[2]}}.$$

Step 7: $\left| F_{j, k}^{[2]} \right|$ to $\left| G_{j, k}^{[2]} \right|$

Now we can easily obtain

$$\left| G_{j, k}^{[2]} \right| \leq C j (1-\beta)^j$$

Step 8: $\left| G_{j, k}^{[2]} \right|$ to $\left| F_{j, k}^{[1]} \right|$

Again, using boundedness of the coefficients of the subdivision scheme $S_j^{[1]}$ one easily obtains

$$\left| F_{j, 2k+s}^{[1]} - F_{j, k+t}^{[1]} \right| \leq C_S C j (1-\beta)^j \quad \text{for } s = 0, 1 \text{ and } t = -1, 0, 1.$$

It follows that $F_{j, k_j}^{[1]}$ constitutes a Cauchy sequence, assuming that $k_j = \lfloor k_{j+1}/2 \rfloor$. By the same argument as in the proof of Lemma 4, there exists a continuous function $\varphi^{[1]}$ which is

the limit of the subdivision scheme $S_j^{[1]}$. As we prove below (see Theorem 9), the bound (9.1) implies that this limit function has Hölder exponent $\log(1 - \beta)/\log \beta - \epsilon$, or more precisely,

$$\left| \varphi^{[1]}(x+t) - \varphi^{[1]}(x) \right| \leq C (1 + |\log |t||) |t|^{\log(1-\beta)/\log \beta}.$$

Also, the $F_{j,k}^{[1]} \equiv f_{j,k}^{[1]}$ are uniformly bounded.

9.2. Convergence and regularity of the original subdivision scheme. Now, we turn our attention to the convergence of $S_j^{[0]}$. The problem here is that the coefficients of this subdivision scheme may become unbounded as the level j goes to infinity, and we cannot apply Lemma 2 directly. It turns out though that we can rearrange the formulas in such a way that all the coefficients stay bounded. We also use this to prove the convergence to a continuous function.

Specifically, we build a piecewise-linear spline $L_j(x)$ for the values $f_{j,k}$ on every level j , that is, we define

$$L_j(x)|_{[x_{j,k}, x_{j,k+1}]} = f_{j,k} + f_{j,k}^{[1]}(x - x_{j,k}).$$

Now we shall prove the uniform convergence of the sequence $L_j(x)$ in $C(K)$, for any compact $K \subset \mathbf{R}$. For this, we only need to consider the differences between the levels at “new” odd points, for all the values at even points do not change from one level to the next one. Namely,

$$\|L_{j+1} - L_j\|_\infty = \sup_k |L_{j+1}(x_{j+1,2k+1}) - L_j(x_{j+1,2k+1})|$$

Let us now consider this in detail in order to obtain the needed estimate.

Without loss of generality consider the scheme at the new point $x_{j+1,3}$. To shorten our notation we again use $[s, t] = x_{j+1,t} - x_{j+1,s}$ consistent with earlier usage, with $k = 0$ now. We have

$$f_{j+1,3} = a_0 f_{j,0} + a_1 f_{j,1} + a_2 f_{j,2} + a_3 f_{j,3},$$

where

$$\begin{aligned} a_0 &= -\frac{[2,3][3,4][3,6]}{[0,2][0,4][0,6]}, & a_1 &= \frac{[0,3][3,4][3,6]}{[0,2][2,4][2,6]}, \\ a_2 &= \frac{[0,3][2,3][3,6]}{[4,6][2,4][0,4]}, & a_3 &= -\frac{[2,3][3,4][0,3]}{[4,6][2,6][0,6]}. \end{aligned}$$

It can be easily seen that terms like $[2,3]/[0,2]$ can become unbounded if $\gamma \rightarrow \infty$. We are interested in getting an estimate in the following form

$$L_{j+1}(x_{j+1,3}) - L_j(x_{j+1,3}) = b_0(f_{j,1} - f_{j,0}) + b_1(f_{j,2} - f_{j,1}) + b_2(f_{j,3} - f_{j,2}).$$

Obviously, $L_j(x_{j+1,3}) = f_{j,1}[3,4]/[2,4] + f_{j,2}[2,3]/[2,4]$. After some algebraic manipulations one gets the following form of the coefficients b_i , $i = 0, 1, 2$

$$\begin{aligned} b_0 &= -a_0 = \frac{1}{[0,2]} \left(\frac{[3,4][3,6]}{[0,4][0,6]} [2,3] \right), \\ b_2 &= a_3 = -\frac{1}{[4,6]} \left(\frac{[2,3][0,3]}{[2,6][0,6]} [3,4] \right), \\ b_1 &= -(a_0 + a_1) \frac{[3,4]}{[2,4]} + (a_0 + a_1) \frac{[3,4]}{[2,4]} = \frac{1}{[2,4]} \left(-\frac{[3,6][3,4]}{[0,6][0,4]} [2,3] + \frac{[0,3][2,3]}{[0,6][2,6]} [3,4] \right). \end{aligned}$$

Hence,

$$|b_0| \leq \frac{1}{[0, 2]}[2, 3], \quad |b_1| \leq 1, \quad |b_2| \leq \frac{1}{[4, 6]}[3, 4].$$

From the uniform boundedness of the first divided differences it follows that

$$|f_{j,1} - f_{j,0}| \leq C_1[0, 2], \quad |f_{j,2} - f_{j,1}| \leq C_1[2, 4], \quad |f_{j,3} - f_{j,2}| \leq C_1[4, 6].$$

Hence,

$$|L_{j+1}(x_{j+1,3}) - L_j(x_{j+1,3})| \leq 3C_1[2, 4]$$

Since $[2, 4]$ is a step of the mesh on the level j we can write

$$\|L_{j+1} - L_j\|_C \leq 3C_1(1 - \beta)^j$$

which means that $\{L_j\}$ is a Cauchy sequence in $C(K)$. Its limit is the limit function $\varphi \equiv \varphi^{[0]}$ of the cubic subdivision scheme. One again has, as in Lemma 5, that $(\varphi)' = \varphi^{[1]}$. We therefore obtain the same regularity result as in the homogeneous case, namely, that φ is C^1 and that

$$|\varphi'(x+t) - \varphi'(x)| \leq C(1 + |\log|t||) |t|^{\log(1-\beta)/\log\beta}.$$

The following theorem combines in one statement all the results of Section 9:

Theorem 9. *The cubic Lagrange interpolation subdivision scheme on the dyadically-balanced mesh converges to a continuously differentiable function $\varphi(y)$. Moreover, $\varphi'(y)$ is Hölder continuous with exponent $\log(1 - \beta)/\log\beta - \epsilon$ ($\epsilon > 0$ arbitrarily small); more precisely*

$$|\varphi'(x+t) - \varphi'(x)| \leq C |t|^{\log(1-\beta)/\log\beta} (1 + |\log|t||).$$

If $\beta \geq 1/3$, then φ' is Hölder continuous with exponent $1 - \epsilon$. More precisely, one has

$$|\varphi'(x+t) - \varphi'(x)| \leq C |t| (1 + |\log|t||). \quad (39)$$

Proof. Applying the results from the preceding sections one can show that $f_{j,k_j(y)}^{[1]}$ constitute a Cauchy sequence with a limit $\varphi'(y)$ which satisfies

$$\left| \varphi'(y) - f_{j,k_j(y)}^{[1]} \right| \leq Cj(1 - \beta)^j$$

Moreover, the following is true

$$\left| f_{j,l+s}^{[1]} - f_{j,l}^{[1]} \right| \leq sC'j(1 - \beta)^j$$

Fix some $t \in \mathbf{R}$, such that $|t| < \min_k d_{0,k}$. There exist some level j so that

$$|k_j(y+t) - k_j(y)| \leq 1 \quad \text{and} \quad |k_{j+1}(y+t) - k_{j+1}(y)| > 1.$$

Note that $|k_{j+1}(y+t) - k_{j+1}(y)| \leq 3$. Then

$$\begin{aligned} |\varphi'(y+t) - \varphi'(y)| &\leq \left| \varphi'(y+t) - f_{j,k_j(y+t)}^{[1]} \right| + \\ &\quad \left| f_{j,k_j(y+t)}^{[1]} - f_{j,k_j(y)}^{[1]} \right| + \left| \varphi'(y) - f_{j,k_j(y)}^{[1]} \right| \leq C''j(1 - \beta)^j \end{aligned}$$

Without loss of generality suppose that $t > 0$. Then, necessarily, $k_{j+1}(y+t) \geq k_j(y)+2$, so that $x_{j+1,k_j(y)} \leq x < x_{j+1,k_j(y)+1} < x_{j+1,k_j(y)+2} \leq x+t$. It follows that $|t| \geq d_{j+1,k_j(y)+1} \geq C\beta^{j+1}$. Finally,

$$|\varphi'(y+t) - \varphi'(y)| \leq C''' |t|^{\log(1-\beta)/\log \beta} (1 + |\log |t||),$$

which concludes the proof of the general case.

If $\beta \geq 1/3$, then we can again sharpen this result (similar to $\gamma \geq \gamma_0$ in the homogeneous case). We then have

$$|G_{j,k}^{[4]}| \leq \frac{C}{D_{j,k}^{[2]}}.$$

The detailed arguments presented earlier can be repeated with slight modifications leading to

$$|F_{j,k}^{[3]}| \leq \frac{C}{D_{j,k}^{[2]}}, \quad |G_{j,k}^{[3]}| \leq C, \quad |F_{j,k}^{[2]}| \leq Cj, \quad |G_{j,k}^{[2]}| \leq CjD_{j,k}^{[2]}.$$

Arguments similar to those above then show that the limit function satisfies

$$|\varphi'(x+t) - \varphi'(x)| \leq C |t| (1 + |\log |t||).$$

□

Note that for the semi-regular case ($\beta = 1/2$) (39) is known to be optimal [14].

10. HIGHER ORDER SCHEMES

So far we have only focused on the cubic case. The next higher order case involves three even neighbors on each side of a new odd point and achieves quintic order. Like in Section 5 we can use the commutation formula to find the derived schemes. After six applications we end up with the subdivision scheme $T_j^{[6]}$ for the $g^{[6]}$. The matrix $T_j^{[6]}$ has the following structure:

$$\begin{array}{l} 2k-5 \\ 2k-4 \\ 2k-3 \\ 2k-2 \\ 2k-1 \end{array} \left[\begin{array}{ccccc} * & * & * & 0 & 0 \\ 0 & \times & \times & 0 & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & * \end{array} \right]$$

$$k-5 \quad k-4 \quad k-3 \quad k-2 \quad k-1$$

The coefficients on row $2k-4$ are given by

$$T_{j,2k-4,k-4}^{[6]} = -\frac{d_{j+1,k-4}^{[6]} d_{j+1,k-6}^{[3]}}{d_{j+1,k-3}^{[2]} d_{j+1,k-3}^{[4]}} \quad \text{and} \quad T_{j,2k-4,k-3}^{[6]} = -\frac{d_{j+1,k-4}^{[6]} d_{j+1,k+1}^{[3]}}{d_{j+1,k-1}^{[2]} d_{j+1,k-3}^{[4]}}$$

while the ones on row $2k - 3$ are given by

$$\begin{aligned}
T_{j,2k-3,k-4}^{[6]} &= \frac{d_{j+1,k-4}^{[1]} d_{j+1,k-6}^{[3]}}{d_{j+1,k-3}^{[2]} d_{j+1,k-3}^{[4]}} \\
T_{j,2k-3,k-3}^{[6]} &= -\frac{d_{j+1,k-4}^{[3]} d_{j+1,k+3}^{[1]}}{d_{j+1,k-1}^{[4]} d_{j+1,k-1}^{[2]}} - \frac{d_{j+1,k-4}^{[8]}}{d_{j+1,k-1}^{[2]}} - \frac{d_{j+1,k+1}^{[3]} d_{j+1,k-4}^{[1]}}{d_{j+1,k-3}^{[4]} d_{j+1,k-1}^{[2]}} \\
T_{j,2k-3,k-2}^{[6]} &= \frac{d_{j+1,k+3}^{[1]} d_{j+1,k+3}^{[3]}}{d_{j+1,k+1}^{[2]} d_{j+1,k-1}^{[4]}}.
\end{aligned}$$

Even without going to a detailed analysis like in the cubic case we can draw some conclusions. In the regular setting the norm of this matrix can be bounded by $11/2$. From this we can conclude that the limit function will be at least $C^{2.54}$. (A more careful analysis for the regular case, using Fourier techniques, shows that the limit function is $C^{2.83}$.) In the homogeneous case one can find a bound for the elements of $T^{[6]}$ which is continuous in γ and coincides with the actual value of the element in case $\gamma = 1$. To do so we need to bound expressions of the type $d_{j,a}^{[A]}/d_{j,b}^{[B]}$. If $a + A = b + 1$ then the intervals $[x_{j,a}, x_{j,a+A}]$ in the numerator and $[x_{j,b}, x_{j,b+B}]$ in the denominator overlap in $[x_{j,a+A-1}, x_{j,a_A}] = [x_{j,b}, x_{j,b+B}]$. Thus $d_{j,a}^{[A]} \leq (1 + \gamma + \gamma^2 + \dots + \gamma^{A-1}) d_{j,b}^{[1]}$ while $d_{j,b}^{[B]} \geq (1 + 1/\gamma + 1/\gamma^2 + \dots + 1/\gamma^{B-1}) d_{j,b}^{[1]}$. If $a + A \neq b + 1$ we use that $d_{j,a}^{[A]} \leq \gamma^l d_{j,a+l}^{[A]}$ to derive that

$$\frac{d_{j,a}^{[A]}}{d_{j,b}^{[B]}} \leq \gamma^{|(a+A)-(b+1)|} \frac{1 + \gamma + \gamma^2 + \dots + \gamma^{A-1}}{1 + 1/\gamma + 1/\gamma^2 + \dots + 1/\gamma^{B-1}}.$$

In case $\gamma = 1$, the bound is equal to the actual value A/B . Using continuity we can see that a region for $\gamma > 1$ exists for which the irregular quintic scheme is at least C^2 .

A similar reasoning holds for all the higher order cases. If we can use the simple estimate above to deduce in the regular case that the limit function is $C^{n+\alpha}$ with $0 < \alpha < 1$, then there exists a region for γ where the irregular scheme is at least C^n . The regions for γ may be quite small, but almost surely this can be improved considerably by detailed analysis.

11. COMMENTS

1. A natural question is whether our approach can be generalized to schemes not derived from interpolating subdivision, e.g., variational schemes. As pointed out before, the difference operators then need to be adapted to the order of the scheme under consideration.
2. Although we did not emphasize this, the subdivision schemes considered in this paper and the associated derived schemes are coupled with a family of dual schemes whose limit functions are non uniform B-splines. We thus implicitly defined an irregularly spaced multiresolution analysis (MRA) and a biorthogonal dual spline MRA. The commutation formula can be used to find the corresponding compactly supported primal and dual wavelet bases. We shall address this in a separate paper.
3. So far we only considered cases where new points are added simultaneously between every pair of adjacent old points. One can also insert new points sequentially, i.e., one at a time.

Then there is a single new point in every round while any other point is old and can be used in the subdivision.

4. The holy grail in this line of work is a deeper understanding of the irregular subdivision in higher dimensions. The two dimensional setting already is much harder because irregular spacing of the points now can be combined with irregular topology of the grid. Several powerful results have been obtained for smooth semi-regular schemes with irregular coarsest level topology [37, 29, 37, 40]. For fully irregular 2D subdivision the field is wide open.

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APPENDIX A. THE HOMOGENEITY CONDITION AND SPACES OF HOMOGENEOUS TYPE.

A metric space (E, d) equipped with a measure μ , is said to be of *homogeneous type* if there exists $C > 0$ so that, for all $x \in E$ and all $R > 0$,

$$\mu(B(x, 2R)) \leq C\mu(B(x, R)), \quad (40)$$

where $B(x, R)$ is the ball with radius R around x . Spaces of homogeneous type were introduced by Coifman and Weiss [7] as a generalization of Euclidian spaces. For many harmonic analysis estimates, (40) is the key ingredient.

A homogeneous multi-level mesh effectively defines a different metric distance d on \mathbf{R} by

$$d(y, z) = \lim_{j \rightarrow \infty} 2^{-j} \# \{k : y \leq x_{j,k} \leq z\}$$

if $y \leq z$, and $d(y, z) = d(z, y)$ if $y > z$. In terms of the function ξ introduced in Section 2, for which $\xi(k2^{-j}) = x_{j,k}$, d can also be written as

$$d(y, z) = |\xi^{-1}(y) - \xi^{-1}(z)|.$$

We want to show that (40) holds true for this metric distance, and for Lebesgue measure.

The following lemma will prove useful:

Lemma 10. *Define $t_1(y, R), t_2(y, R)$, and $s_1(y, R), s_2(y, R)$ by*

$$t_j(y, R) = \xi(y + jR) - \xi(y), \quad s_j(y, R) = \xi(y) - \xi(y - jR).$$

Then there exists a constant C , independent of y or R , so that

$$t_2(y, R) \leq Ct_1(y, R), \quad s_2(y, R) \leq Cs_1(y, R).$$

Proof. For $R < 4$, find $j \geq 0$, so that $2^{-j+1} \leq R < 2^{-j+2}$. Find l so that $(l-1)2^{-j} \leq y \leq l2^{-j}$. Since $(l+1)2^{-j} - y \leq 2^{-j+1} \leq R$, both $l2^{-j}$ and $(l+1)2^{-j} \in [y, y+R]$. On the other hand, $(l+8)2^{-j} - y \geq 2^{-j+3} > 2R$, so that $(l+8)2^{-j} \notin [y, y+2R]$. It follows that

$$t_2(y, R) = \xi(y+2R) - \xi(y) < x_{j,l+8} - x_{j,l-1} = \sum_{k=l-1}^{l+7} d_{j,k},$$

whereas $t_1(y, R) = \xi(y+R) - \xi(y) \geq x_{j,l+1} - x_{j,l} = d_{j,l}$. By homogeneity, $t_2(y, R)/t_1(y, R) \leq \gamma + \sum_{k=0}^7 \gamma^k$.

For $R \geq 4$, find n, l so that $l-1 \leq y \leq l$, $n-1 \leq R < n$. Then

$$\begin{aligned} t_2(y, R) &< x_{0,l+2n} - x_{0,l-1} \leq (2n+1) \max_l d_{0,l}, \\ t_1(y, R) &\geq x_{0,l+n-2} - x_{0,l} \geq (n-2) \min_l d_{0,l}, \end{aligned}$$

so that

$$t_2(y, R)/t_1(y, R) \leq \frac{2n+1}{n-2} \frac{\max_l d_{0,l}}{\min_l d_{0,l}} \leq \frac{11}{3} \frac{\max_l d_{0,l}}{\min_l d_{0,l}},$$

since $n \geq 5$.

It now suffices to take

$$C = \max \left\{ \gamma + \sum_{k=0}^7 \gamma^k, \frac{11}{3} \frac{\max_l d_{0,l}}{\min_l d_{0,l}} \right\}$$

to prove our claim for t_1, t_2 . The proof for s_1, s_2 is analogous. □

Let us now look at $\mu(B(x, jR))$. We have

$$B(x, jR) = [x^-(jR), x^+(jR)],$$

where $x^\sigma(jR) = \xi(\xi^{-1}(x) + \sigma jR)$, for $\sigma = +$ or $-$. So

$$\begin{aligned} \mu(B(x, jR)) &= x^+(jR) - x^-(jR) = \xi(\xi^{-1}(x) + jR) - \xi(\xi^{-1}(x) - jR) = \\ &= t_j(\xi^{-1}(x), R) + s_j(\xi^{-1}(x), R). \end{aligned}$$

The estimate (40) now immediately follows from Lemma 10.

If the multi-level mesh is merely dyadically balanced but not homogeneous, then we can still define the metric distance d , but (\mathbf{R}, d) need not be of homogeneous type. Let us, for

instance, revisit the example $x_{0,l} = l \in \mathbf{Z}$, $x_{j+1,2l+1} = x_{j,l} + d_{j,l}/3$, where $\beta = 1/3$. Then $\xi(2^{-j}) = (1/3)^j$, $\xi(-2^{-j}) = (2/3)^j$, so that $\mu(B(3^{-j}, 2^{-j})) = \xi(2^{-j+1}) - \xi(0) = 3^{-j+1}$, but $\mu(B(3^{-j}, 2^{-j+1})) = \xi(3 \cdot 2^{-j}) - \xi(-2^j) > (2/3)^j$. Thus $\mu(B(3^{-j}, 2^{-j+1}))/\mu(B(3^{-j}, 2^{-j})) > 2^j/3$, and no uniform bound of type (40) exists.

APPENDIX B. PROOF THAT $C_j = 1$

It follows from (9) that

$$S_{j,l,m}^{[p-1]} = \sum_{n=-\infty}^{l-1} d_{j+1,n}^{[p]} \left(\frac{S_{j,n,m-1}^{[p]}}{d_{j,m-1}^{[p]}} - \frac{S_{j,n,m}^{[p]}}{d_{j,m}^{[p]}} \right) = \frac{1}{d_{j,m-1}^{[p]}} \sum_{n=-\infty}^{l-1} d_{j+1,n}^{[p]} S_{j,n,m-1}^{[p]} - \frac{1}{d_{j,m}^{[p]}} \sum_{n=-\infty}^{l-1} d_{j+1,n}^{[p]} S_{j,n,m}^{[p]}.$$

Note that both sums are finite, as all other sums in this appendix. Now

$$\sum_{n=-\infty}^{l-1} d_{j+1,n}^{[p]} S_{j,n,m-1}^{[p]} + \sum_{n=l}^{+\infty} d_{j+1,n}^{[p]} S_{j,n,m-1}^{[p]} = C_j d_{j,m-1}^{[p]}.$$

Hence, we have

$$\sum_{n=-\infty}^{l-1} d_{j+1,n}^{[p]} S_{j,n,m-1}^{[p]} = C_j d_{j,m-1}^{[p]} - \sum_{n=l}^{+\infty} d_{j+1,n}^{[p]} S_{j,n,m-1}^{[p]}.$$

Then

$$S_{j,l,m}^{[p-1]} = C_j - \frac{1}{d_{j,m-1}^{[p]}} \sum_{n=l}^{+\infty} d_{j+1,n}^{[p]} S_{j,n,m-1}^{[p]} - \frac{1}{d_{j,m}^{[p]}} \sum_{n=-\infty}^{l-1} d_{j+1,n}^{[p]} S_{j,n,m}^{[p]}.$$

Now, we want the scheme $S_j^{[p-1]}$ to preserve constants, therefore

$$\sum_{m=m_1(l)}^{m_2(l)} S_{j,l,m}^{[p-1]} = 1.$$

On the other hand

$$\begin{aligned} \sum_{m=m_1(l)}^{m_2(l)} S_{j,l,m}^{[p-1]} &= \sum_{m=m_1(l)}^{m_2(l)} \left(C_j - \frac{1}{d_{j,m-1}^{[p]}} \sum_{n=l}^{+\infty} d_{j+1,n}^{[p]} S_{j,n,m-1}^{[p]} - \frac{1}{d_{j,m}^{[p]}} \sum_{n=-\infty}^{l-1} d_{j+1,n}^{[p]} S_{j,n,m}^{[p]} \right) = \\ &= [m_2(l) - m_1(l) + 1] C_j - \frac{1}{d_{j,m_1(l)-1}^{[p]}} \sum_{n=l}^{\infty} d_{j+1,n}^{[p]} S_{j,n,m_1(l)-1}^{[p]} - \frac{1}{d_{j,m_2(l)}^{[p]}} \sum_{n=-\infty}^{l-1} d_{j+1,n}^{[p]} S_{j,n,m_2(l)}^{[p]} - \\ &= \sum_{m=m_1(l)}^{m_2(l)-1} \frac{1}{d_{j,m}^{[p]}} \sum_{n=-\infty}^{+\infty} d_{j+1,n}^{[p]} S_{j,n,m}^{[p]} = [m_2(l) - m_1(l) + 1] C_j - 0 - 0 - \sum_{m=m_1(l)}^{m_2(l)-1} C_j = C_j. \end{aligned}$$

Hence, $C_j = 1$.

APPENDIX C. PROOF OF LEMMA 5.

Proof. The sequences $\{f_{j,k_j(y)}^{[p]}\}$ converge uniformly to uniformly continuous functions $\varphi^{[p]}(y)$ for $p = 1, \dots, P-1$. Consequently, these sequences are uniformly bounded.

The only thing we need to prove is that, for $p = 1, \dots, P-1$, we have

$$p \varphi^{[p]}(y) = \frac{d\varphi^{[p-1]}(y)}{dy}.$$

We will prove that

$$\lim_{t \rightarrow 0} \frac{\varphi^{[p-1]}(y+t) - \varphi^{[p-1]}(y)}{t} = p \varphi^{[p]}(y).$$

Fix arbitrarily small $\epsilon > 0$. There exist $\delta > 0$ so that

$$\left| \varphi^{[p]}(x) - \varphi^{[p]}(y) \right| < \epsilon/p \quad \text{for all } x \in (y - \delta, y + \delta).$$

Now fix arbitrary $|t| < \delta$. There exist $j = j(t)$ so that all of the following is true for any $x \in (y - \delta, y + \delta)$:

$$\left| \varphi^{[p]}(y) - f_{j,k_j(y)}^{[p]} \right| < \epsilon/p, \quad (41)$$

$$\left| \varphi^{[p-1]}(x) - f_{j,k_j(x)}^{[p-1]} \right| < \epsilon|t|, \quad (42)$$

$$\left| y - \frac{x_{j,k_j(y)} + \dots + x_{j,k_j(y)+p-1}}{p} \right| < \min \left\{ \frac{\epsilon t^2}{M_{[p-1]}}, \frac{|t|}{4} \right\}, \quad (43)$$

$$\left| y+t - \frac{x_{j,k_j(y+t)} + \dots + x_{j,k_j(y+t)+p-1}}{p} \right| < \min \left\{ \frac{\epsilon t^2}{M_{[p-1]}}, \frac{|t|}{4} \right\}, \quad (44)$$

where $M_{[p-1]} = \sup_{j,k} \left| f_{j,k}^{[p-1]} \right| < \infty$.

Introduce the following notation:

$$\Delta_t \varphi^{[p-1]} = \varphi^{[p-1]}(y+t) - \varphi^{[p-1]}(y)$$

$$\Delta_t f^{[p-1]} = f_{j,k_1}^{[p-1]} - f_{j,k_0}^{[p-1]}, \quad \text{where } k_0 = k_j(y) \text{ and } k_1 = k_j(y+t)$$

$$\Delta_t x = \frac{x_{j,k_1} + \dots + x_{j,k_1+p-1}}{p} - \frac{x_{j,k_0} + \dots + x_{j,k_0+p-1}}{p}.$$

Then from (42)-(44)

$$\begin{aligned} \left| \frac{\Delta_t \varphi^{[p-1]}}{t} - p \varphi^{[p]}(y) \right| &\leq \left| \frac{\Delta_t \varphi^{[p-1]}}{t} - \frac{\Delta_t f^{[p-1]}}{t} \right| + \left| \frac{\Delta_t f^{[p-1]}}{t} - \frac{\Delta_t f^{[p-1]}}{\Delta_t x} \right| + \left| \frac{\Delta_t f^{[p-1]}}{\Delta_t x} - p \varphi^{[p]}(y) \right| \\ &\leq 2\epsilon + 4\epsilon + \left| \frac{\Delta_t f^{[p-1]}}{\Delta_t x} - \frac{\varphi^{[p]}(y)}{p} \right|. \end{aligned}$$

From the definition of divided differences we have

$$\Delta_t f^{[p-1]} = f_{j,k_1}^{[p-1]} - f_{j,k_0}^{[p-1]} = \sum_{k=k_0}^{k_1-1} (f_{j,k+1}^{[p-1]} - f_{j,k}^{[p-1]}) = \sum_{k=k_0}^{k_1-1} f_{j,k}^{[p]} (x_{j,k+p} - x_{j,k}).$$

Using $\sum_{k=k_0}^{k_1-1} (x_{j,k+p} - x_{j,k}) = p \Delta_t x$, together with (41), we obtain

$$\left| \frac{\Delta_t f^{[p-1]}}{\Delta_t x} - p\varphi^{[p]}(y) \right| \leq \epsilon.$$

It follows that for any $\epsilon > 0$ there exists $\delta > 0$ so that for all $t \in (-\delta, \delta), t \neq 0$ we have

$$\left| \frac{\varphi^{[p-1]}(y+t) - \varphi^{[p-1]}(y)}{t} - p\varphi^{[p]}(y) \right| < 7\epsilon.$$

Hence

$$p\varphi^{[p]}(y) = \lim_{t \rightarrow 0} \frac{\varphi^{[p-1]}(y+t) - \varphi^{[p-1]}(y)}{t} = \frac{d\varphi^{[p-1]}(y)}{dy},$$

which concludes the proof. \square

APPENDIX D.

In Section 7.2, we claim that for γ not exceeding some γ_0 , we have $G_{j,k}^{[4]} \leq C/D_{j,k}^{[2]}$ for cubic Lagrange interpolation. In order to prove this, it is sufficient to show that (with the notations introduced at the start of Section 7)

$$\frac{[-2, -1][0, 2]}{[-1, 1][-2, 2]} + \frac{[0, 2][3, 4]}{[1, 3][0, 4]} \leq 1 \quad (45)$$

if $\gamma \leq \gamma_0$.

Let us estimate the left hand side of (45). Writing $d_r, r = 1, \dots, 6$, for $[-2+r-1, -2+r]$, we need to estimate

$$\frac{d_1(d_3 + d_4)}{(d_2 + d_3)(d_1 + d_2 + d_3 + d_4)} + \frac{d_6(d_3 + d_4)}{(d_4 + d_5)(d_3 + d_4 + d_5 + d_6)} \quad (46)$$

under the constraints that $\gamma^{-1}d_r \leq d_{r+1} \leq \gamma d_r$, for $r = 1, \dots, 5$. Using $d_1 \leq \gamma d_2$, $d_6 \leq \gamma d_5$, and the monotonicity of $d/(A+d)$, we obtain

$$(46) \leq \gamma \left(\frac{d_2(d_3 + d_4)}{(d_2 + d_3)((1+\gamma)d_2 + d_3 + d_4)} + \frac{d_5(d_3 + d_4)}{(d_4 + d_5)(d_3 + d_4 + (1+\gamma)d_5)} \right). \quad (47)$$

Next, we use that $d[(A+d)(B+d)]^{-1}$ achieves its maximum $(\sqrt{A} + \sqrt{B})^{-2}$ at $d = \sqrt{AB}$, so that

$$\begin{aligned} (47) &\leq \frac{\gamma}{1+\gamma} \left(\frac{d_3 + d_4}{(\sqrt{d_3} + \sqrt{\frac{d_3+d_4}{1+\gamma}})^2} + \frac{d_3 + d_4}{(\sqrt{d_4} + \sqrt{\frac{d_3+d_4}{1+\gamma}})^2} \right) \\ &\leq \gamma \left(\frac{1}{\left(\sqrt{\frac{d_3}{d_3+d_4}(1+\gamma)} + 1\right)^2} + \frac{1}{\left(\sqrt{\frac{d_4}{d_3+d_4}(1+\gamma)} + 1\right)^2} \right). \end{aligned} \quad (48)$$

Introducing $a = \sqrt{1+\gamma}$, $\cos^2 \theta = d_3/(d_3 + d_4)$, the right hand side of (48) can be rewritten as

$$\frac{\gamma}{(1+a \cos \theta)^2} + \frac{\gamma}{(1+a \sin \theta)^2}.$$

As a function of θ , this function has a unique minimum on $[0, \pi/2]$ at $\theta = \pi/4$; it achieves its maximum at the edges $\theta = 0$ or $\pi/2$. In our case $\cos^2 \theta$ is constrained to lie between $1/(1 + \gamma)$ and $\gamma/(1 + \gamma)$, and the maximum is then attained at these points. So we obtain

$$(48) \leq \gamma \left(\frac{1}{(\sqrt{\gamma} + 1)^2} + \frac{1}{4} \right). \quad (49)$$

The right hand side of (49) is monotone increasing in γ ; it is equal to 1 for $\gamma = \gamma_0 \approx 2.4992$. It follows that (45) will hold for $\gamma \leq \gamma_0$. We also tried to add the constraints coming from homogeneity on the coarser level, $\gamma^{-1}(d_r + d_{r+1}) \leq (d_{r+2} + d_{r+3}) \leq \gamma(d_r + d_{r+1})$ for $r = 1, 3$, and used AMPL, a non-linear optimization package, to find the maximum. However, adding these constraints does not lead to a larger γ_0 .