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Interpolation through an Iterative Scheme

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A new method of interpolation is generated through an iterative scheme on dyadic rationals. The process is linear, local and produces almost twice differentiable functions. A fundamental interpolating function is analyzed. Error formulas are derived. The method is applied for curve fitting. © 1986 Academic Press, Inc.

We want to describe a new method of interpolation which is linear and which produce almost twice differentiable functions. Given a function $y(n)$ defined on the set of relative integers, we want to extend smoothly this function to the real axis. First, let D_n be the set of dyadic rationals $m/2^n$, where m is an arbitrary relative integer, $y(t)$ is already defined on D_0. By induction, one extends y to D_1, D_2, D_3,.... If y is already extended to D_n, if $h = 2^{-n-1}$ and if t belongs to D_{n+1} but not to D_n, the numbers $t-3h$, $t-h$, $t+h$, and $t+3h$ belongs to D_n and we set

$$y(t) = [-y(t-3h) + 9y(t-h) + 9y(t+h) - y(t+3h)]/16. \qquad (1)$$

The extension $y(t)$ has been done to the set D of dyadic rationals p/q, where p is a relative integer, q is a power of 2.

We summarize the main results we will prove. In the first sections, basic properties of the interpolating scheme are presented. If the original sequence comes from the values of a cubic polynomial $p(n)$, then $y(t) = p(t)$. In Section 2, the fundamental function $F(t)$ is introduced, $F(t)$ is the interpolation of he sequence $F(0) = 1$ and $F(n) = 0$ elsewhere. F vanishes outside $(-3, 3)$. Any interpolation function $y(t)$ is linear combination of translates of F, $F(t-n)$. In Section 3, we indicate how to compute $y(t)$ with small memory requirement.

In next sections, the study of differentiability is carried through after an analysis of finite differences of order two. The function y already defined on D has a continuous extension to the real axis; of course, this extension is unique and will be called again by $y(t)$. y is continuously differentiable. The function y' is close to be differentiable: for each positive integer N, there is a number C_N such that for any positive h and for any t of $[-N, N]$,

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$|y'(t+h)-y'(t)| \leqslant C_N h \log(1/h)$. For most sequences $y(n)$, this inequality cannot be improved. In Section 6, we find a simple formula for the value $y'(t)$ when t is a dyadic number. For example $y'(0)=[y(-2)-8y(-1)+8y(1)-y(2)]/12$.

In Section 7, we will find a Fourier expansion for the function $y(t)$ when the original sequence is exponential. This is useful for periodic interpolation. In Section 8, other properties of the fundamental interpolation F are investigated. F, an even function, is positive on $(-1, 1)$, negative in $(1, 2)$. In $(2, 3)$, highly damped oscillations occur. Many functional relations for F are derived. The topics of Section 9 are errors that arise when a function f is replaced with the interpolation $y(t)$ corresponding to the sequence $f(n)$. Interpolation within a finite interval from a finite sequence is considered in Section 10. We conclude with an application of the interpolation scheme to curve fitting in the plane.

1. Basic Properties of the Interpolation

The interpolation according to the scheme (1) of sequences $y(n)$ by functions $y(t)$ defined over D satisfies some simple properties. The interpolation process is a linear process. If $y_1(n)$ and $y_2(n)$ are two functions defined on the relative integers and if $y(n)=y_1(n)+y_2(n)$, then the respective interpolations $y_1(t)$, $y_2(t)$, and $y(t)$ defined on dyadic numbers D satisfy the linear relation: $y(t)=y_1(t)+y_2(t)$. Likewise, for any real scalar c, the interpolation corresponding to $cy(n)$ is $cy(t)$. Another basic property is the translation property of the interpolation. If $y(n)$ is a given sequence with $y(t)$ as interpolation and if m is a relative integer, the interpolation of the sequence $y(n+m)$ is the function $y(t+m)$. We introduce a definition we will often use.

Definition. If t is a dyadic number, we defines the depth of t as the first integer k for which t belongs to D_k.

Theorem 1. *If $p(t)$ is a polynomial whose degree does not exceed 3, if $y(n)=p(n)$, then the interpolation of the sequence $y(n)$ is precisely $y(t)=p(t)$.*

Proof. The simplest thing to do is to use an induction on the depth of t. If the depth of t is 0, then t is a relative integer and $y(t)=p(t)$. If the depth of t is n and greater than 0, we set $h=2^{-n}$, we know that

$$\begin{aligned} y(t) &= [-y(t-3h)+9y(t-h)+9y(t+h)-y(t+3h)]/16 \\ &= [-p(t-3h)+9p(t-h)+9p(t+h)-p(t+3h)]/16 \end{aligned}$$

by hypothesis of induction since the depths of $t-3h$, $t-h$, $t+h$, and $t+3h$ are smaller than n.

We use the following Lagrange polynomials.

$$L_0(u)=(-1-u)(1-u)(3-u)/48, \qquad L_1(u)=(3+u)(1-u)(3-u)/16,$$

$$L_2(u)=(3+u)(1+u)(3-u)/16, \qquad L_3(u)=(3+u)(1+u)(-1+u)/48.$$

If f is a cubic polynomial, $f(0)=\sum_{k=0}^{3} f(2k-3)\, L_k(0)=[-f(-3)+9f(-1)+9f(1)-f(3)]/16$. We set $f(u)=p(t+uh)$. Applying the previous formula, we get $y(t)=p(t)$.

THEOREM 2. *If $y(t)$ is the interpolation of the sequence $y(n)$ defined over D_0, if h is a negative integral power of 2, then the interpolation of the sequence $y(nh)$ is the function $y(th)$.*

The proof is almost trivial, but the homogeneity of the process of interpolation is important.

2. THE FUNDAMENTAL INTERPOLATION

Let us start with the following sequence $y(0)=1$ and $y(n)=0$ when n is any other integer. Using the scheme (1), this sequence can be extended to D, the dyadic numbers. Let us denote by $F(t)$ this extension, F will be called the fundamental interpolation.

LEMMA 3. *The fundamental interpolation vanishes outside* $(-3, 3)$.

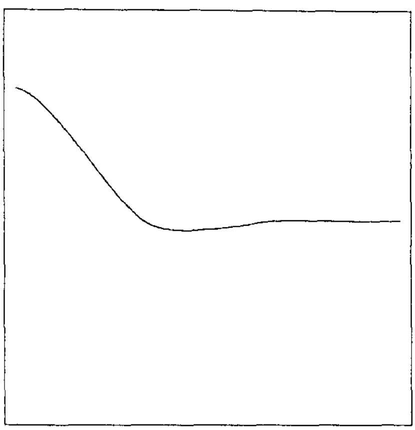

FIG. 1. Graph of the fundamental function over [0, 3].

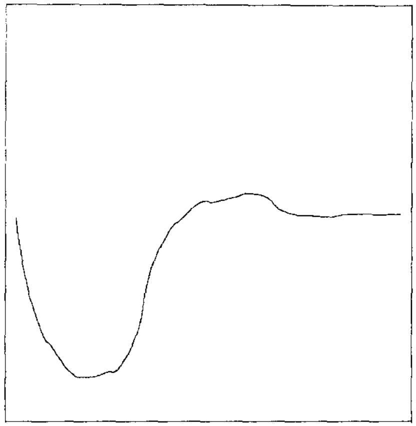

FIG. 2. Derivative of the fundamental function on [0, 3].

Proof. We set $t_0 = 0$ and t_n is defined with the recurrence $t_{n+1} = t_n + 3.2^{-n-1}$. As easily checked, the restriction of F to D_n vanishes outside $[-t_n, t_n]$. Since all these intervals are inside $(-3, 3)$, F vanishes outside $(-3, 3)$.

THEOREM 4. *If $y(n)$ is a function on D_0, the extension $y(t)$ to D according to the scheme* (1) *is $\sum_{n=k-2}^{k+3} y(n)\, F(t-n)$, where k is the integral part of t.*

Proof. By Lemma 3, $\sum_{n=k-2}^{k+3} y(n)\, F(t-n) = \sum_{n=-\infty}^{\infty} y(n)\, F(t-n)$. Since the interpolating scheme (1) is linear, the last series is in fact the interpolation for the sequence $y(n)$.

We will see later that F is uniformly differentiable. Figures 1 and 2 represent the graph of F and F' over [0, 3].

3. FLOW OF COMPUTATIONS

Let $y(n)$ be a sequence defined on the relative integers D_0, we use again $y(t)$ as the extension to the set D of dyadic numbers according to the scheme (1). If t is a dyadic number of depth n, one natural way for the computation of $y(t)$ is to introduce a matrix $(n+1) \times 6$: $A = a(i, j)$, $i = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, 6$. The ith row of the matrix A is defined as follows: if m is the integral part of $t2^i$, then $a(i, j) = y([m-3+j]\, 2^{-i})$, $j = 1, 2, \ldots, 6$. Each successive row can be computed from the preceding one.

If the fractional part of $t2^{i+1}$ is less than $\frac{1}{2}$, then

$$a(i+1, 1) = a(i, 2),$$
$$a(i+1, 2) = [-a(i, 1) + 9a(i, 2) + 9a(i, 3) - a(i, 4)]/16,$$
$$a(i+1, 3) = a(i, 3),$$
$$a(i+1, 4) = [-a(i, 2) + 9a(i, 3) + 9a(i, 4) - a(i, 5)]/16,$$
$$a(i+1, 5) = a(i, 4),$$
$$a(i+1, 6) = [-a(i, 3) + 9a(i, 4) + 9a(i, 5) - a(i, 6)]/16.$$

If the fractional part of $t2^{i+1}$ is not less than 1/2, then

$$a(i+1, 1) = [-a(i, 1) + 9a(i, 2) + 9a(i, 3) - a(i, 4)]/16,$$
$$a(i+1, 2) = a(i, 3),$$
$$a(i+1, 3) = [-a(i, 2) + 9a(i, 3) + 9a(i, 4) - a(i, 5)]/16,$$
$$a(i+1, 4) = a(i, 4),$$
$$a(i+1, 5) = [-a(i, 3) + 9a(i, 4) + 9a(i, 5) - a(i, 6)]/16,$$
$$a(i+1, 6) = a(i, 5).$$

Of course $y(t)$ is one definite element of the last row of A. The computation of $y(t)$ does not require more than $6n$ memory locations. To compute $y(t)$ for increasing values of t in $D_n \cap [0, 1]$ and to send values $y(t)$ to an external file, suitable updates of A will do the task with almost the same amount of memory.

4. Study of Finite Difference of Order 2

If $y(n)$ is a given function defined on D_0, we still interpolate $y(t)$ over D using the scheme (1). We want to study this function $y(t)$ when t is between 0 and 1. Basic properties of $y(t)$ will come from the knowledge of finite differences of order 2 of y. We define $u_n(t)$ as $[y(t-h) - 2y(t) + y(t+h]/h^2$, where $h = 2^{-n}$. If the depth of t is d, then we set $v(t) = u_{d+1}(t) - u_d(t)$.

Theorem 5. *If the depth of the dyadic number t is n and if $h = 2^n$, then*

$$u_n(t) = [u_{n-1}(t-h) + (u_{n-1}(t+h)]/2 \qquad \text{when} \quad n > 0 \qquad (2)$$

$$v(t) = -[v(t-h) + v(t+h)]/4, \qquad (3)$$

$$u_m(t) = u_n(t) + (m-n)\, v(t) \qquad \text{when} \quad m \geqslant n. \qquad (4)$$

Proof. Let us check the first statement. Let us denote by a, b, c, d and e the successive values of the function y at the points $t-3h$, $t-h$, t, $t+h$, $t+3h$. These points with the exception of t belong to D_{n-1}; so, by definition of $y(t)$: $c=(-a+9b+9d-e)/16$. The value $u_n(t)$ is $(b-2c+d)/h^2 = (a-b-d+e)/(8h^2) = [(a-2b+d)+(b-2d+e)]/(8h^2)$. This last value is precisely $[u_{n-1}(t-h)+(u_{n-1}(t+h)]/2$.

Let us go on for the second statement with the same notation. Let $f=y(t-2h)$ and $g=y(t+2h)$, the depth of $t-h/2$ and $t+h/2$ is $n+1$. $y(t-h/2)=(-f+9b+9c-d)/16$ and $y(t+h/2)=(-b+9c+9d-g)/16$. So $u_{n+1}(t)=(-f+8b-14c+8d-g)/(4h^2)$.

$$\begin{aligned}v(t)&=u_{n+1}(t)-u_n(t)=u_{n+1}(t)-(b-2c+d)/h^2\\&=(-f+4b-6c+4d-g)/(4h^2)\\&=[-(f-2b+c)+2(b-2c+d)-(c-2d+g)]/(4h^2)\\&=[-u_n(t-h)+2u_n(t)-u_n(t+h)]/4\\&=[-u_{n-1}(t-h)-v(t-h)+2u_n(t)-u_{n-1}(t+h)-v(t+h)]/4.\end{aligned}$$

According to formula (2), this reduces to $-[v(t-h)+v(t+h)]/4$.

To prove statement (4), let us change the previous notation. If the depth of t is n without restriction on the value of n, if m is greater than $n+1$, we set $h=2^m$, we mean by a, b, c, d and e the values of the function y at the respective points $t-4h$, $t-2h$, t, $t+2h$, and $t+4h$. Since t belongs to D_n and $m>n$, then $t-h$ and $t+h$ do not belong to D_{m-1}. We can evaluate y at this to points,

$$\begin{aligned}y(t-h)&=(-a+9b+9c-d)/16 \qquad \text{and}\\y(t+h)&=(-b+9c+9d-e)/16;\\u_m(t)&=[y(t-h)-2y(t)+y(t+h)]/h^2\\&=(-a+8b-14c+8d-e)/(16h^2)\\&=-(a-2c+e)/(16h^2)+(b-2c+d)/(2h^2)\\&=-u_{m-2}(t)+2u_{m-1}(t),\\u_m(t)-u_{m-1}(t)&=u_{m-1}(t)-u_{m-2}(t).\end{aligned}$$

Formula (4) follows from that identity.

THEOREM 6. *If $y(n)$ is a sequence for interpolation, let a be the maximum of the absolute values of the central differences of order* 2 *computed at* 0 *and* 1 *and let b be the maximum of the absolute values of the central differences* *of order* 4 *computed at* 0 *and* 1, *then for any* t *between* 0 *and* 1 *for any integer* n *greater than the depth of* t, $|u_n(t)| \leqslant a + nb/4$.

Proof. $u_0(0) = y(-1) - 2y(0) + y(1)$ and $u_0(1) = y(0) - 2y(1) + y(2)$; a is equal to the maximum between $|u_0(0)|$ and $|u_0(1)|$,

$$\begin{aligned} v(0) = u_1(0) - u_0(0) &= 4[y(-1/2) - 2y(0) + y(1/2)] - u_0(0) \\ &= [-y(-2) + 8y(-1) - 14y(0) + 8y(1) - y(2)]/4 - u_0(0) \\ &= [-y(-2) + 4y(-1) - 6y(0) + 4y(1) - y(2)]/4. \end{aligned}$$

This is $-1/4$ times the central difference of order 4 computed at 0; b is equal to the maximum between $4\,|v(0)|$ and $4\,|v(1)|$.

From formula (3), it follows that for any dyadic number of $[0, 1]$, $|v(t)|$ is bounded by $b/4$. The inequality $|u_n(t)| \leqslant a + nb/4$ can be proven by a simple induction on n by using formulas (2) and (4).

5. Continuity and Differentiability

We prove in this section that the function $y(t)$ defined for dyadic numbers can be extended continuously to the real axis, still called $y(t)$. $y(t)$ is then a differentiable function. We will describe also the modulus of continuity of $y'(t)$. In this section, we will assume that the variable t varies from 0 to 1, just for the sake of simplicity. From Theorem 6, we bring back the two parameters a and b.

Lemma 7. *If* t *is a dyadic number of* $[0, 1) \cap D_m$, *where* $m > n \geqslant 0$, *if* $h = 2^{-m}$, $h' = 2^{-n}$ *and if* t' *is* h' *times the integral part of* $2^n t$, *then* $|[y(t' + h') - y(t')]/h' - [y(t+h) - y(t)]/h| \leqslant [a + (n + 2)\,b/4]\,2^{-n}/2$.

Proof. We define the sequence of points $\{t_r\}_{r=n}^{m}$: t_r is 2^{-r} times the integral part of $2^r t$. $t_m = t$ and $t_n = t'$. We set $h_r = 2^{-r}$. For each value of r between $n+1$ and m, we get

$$\begin{aligned} &|[y(t_{r-1} + h_{r-1}) - y(t_{r-1})]/h_{r-1} - [y(t_r + h_r) - y(t_r)]/h_r| \\ &\quad = |u_r(t_{r-1} + h_r)\,h_r/2| \leqslant [a + rb/4]\,2^{-r}/2. \end{aligned}$$

The last inequality comes from Theorem 6. $[y(t' + h') - y(t')]/h' - [y(t+h) - y(t)]/h$ is the sum

$$\sum_{r=n+1}^{m} [y(t_{r-1} + h_{r-1}) - y(t_{r-1})]/h_{r-1} - [y(t_r + h_r) - y(t_r)]/h_r.$$

From the triangular inequality, it follows that

$$
\begin{aligned}
&|[y(t'+h')-y(t')]/h' - [y(t+h)-y(t)]/h| \\
&\quad \leqslant \sum_{r=n+1}^{m} [a+rb/4]\, 2^{-r}/2 \\
&\quad \leqslant \sum_{r=n+1}^{\infty} [a+rb/4]\, 2^{-r}/2 = [a+(n+2)\,b/4]\, 2^{-n}/2.
\end{aligned}
$$

This is the required inequality.

LEMMA 8. *If s and t are two numbers of $[0,1)\cap D_m$, if $n \leqslant m$, if $|t-s| \leqslant 2^{-n}$ and if $h = 2^{-m}$, then*

$$
\begin{aligned}
&|[y(t+h)-y(t)]/h - [y(s+h)-y(s)]/h| \\
&\quad \leqslant [2a+(n+1)\,b/2]\, 2^{-n}.
\end{aligned}
$$

Proof. Let t' be the largest value of D_n smaller or equal to t, similarly s' is the largest value of D_n smaller or equal to s. If $h' = 2^{-n}$, then according to the previous lemma,

$$
\begin{aligned}
&|[y(t'+h')-y(t')]/h' - [y(t+h)-y(t)]/h| \\
&\quad \leqslant [a+(n+2)\,b/4]\, 2^{-n}/2, \\
&|[y(s'+h')-y(s')]/h' - [y(s+h)-y(s)]/h| \\
&\quad \leqslant [a+(n+2)\,b/4]\, 2^{-n}/2, \\
&|[y(t+h)-y(t)]/h - [y(s+h)-y(s)]/h| \\
&\quad = |[y(t+h)-y(t)]/h - [y(t'+h')-y(t')]/h' \\
&\qquad + [y(t'+h')-y(t')]/h' \\
&\qquad - [y(s'+h')-y(s')]/h' + [y(s'+h')-y(s'-)]/h' \\
&\qquad - [y(s+h)-y(s)]/h|, \\
&|[y(t+h)-y(t)]/h - [y(s+h)-y(s)]/h| \\
&\quad \leqslant [a+(n+2)\,b/4]\, 2^{-n} \\
&\qquad + |[y(t'+h')-y(t')]/h' - [y(s'+h')-y(s')]/h'|.
\end{aligned}
$$

$|t'-s'|$ is either 0 or h', and $|[y(t'+h')-y(t')]/h' - [y(s'+h')-y(s')]/h'|$ is either 0 or $|u_n(t+h')|h'$. From Theorem 6, it follows that

$$|[y(t'+h')-y(t')]/h'-[y(s'+h')-y(s')]/h'|$$
$$\leqslant (a+nb/4)\,2^{-n},$$
$$|[y(t+h)-y(t)]/h-[y(s+h)-y(s)]/h|$$
$$\leqslant [2a+(n+1)\,b/2]\,2^{-n}.$$

Lemma 9. *If t_1, t_2, t_3, and t_4 are four dyadic numbers of* $[0, 1]$ *such that $t_1 < t_2 \leqslant t_4$, $t_1 \leqslant t_3 < t_4$ and $t_4 - t_1 \leqslant 2^{-n}$, then*

$$|[y(t_2)-y(t_1)]/(t_2-t_1)-[y(t_4)-y(t_3)]/(t_4-t_3)|$$
$$\leqslant [2a+(n+1)\,b/2]\,2^{-n}.$$

Proof. There is an integer m for which the four numbers t_1, t_2, t_3, and t_4 belong to D_m. If $h=2^{-m}$, we set $M=(t_2-t_1)/h$ and $N=(t_4-t_3)/h$. The differential quotient $[y(t_2)-y(t_1)]/(t_2-t_1)$ is the arithmetical mean of M more elementary differential quotients $[y(s+h)-y(s)]/h$. The differential quotient $[y(t_4)-y(t_3)]/(t_4-t_3)$ is the arithmetical mean of N more elementary differential quotients $[y(t+h)-y(t)]/h$. Each couple of elementary differential quotients are close by $[2a+(n+1)\,b/2]\,2^{-n}$. The same is true for the two original differential quotients.

THEOREM 10. *Given a sequence $y(n)$ defined on the relative integers, the interpolation $y(t)$ according to the scheme* (1) *is uniformly continuous on* $D \cap [0, 1]$ *and has a unique extension to the real axis. This extension, still called $y(t)$, is uniformly differentiable on* $[0, 1]$. *If $t \in [0, 1]$, $|y'(t)-y(1)+y(0)| \leqslant a/2+b/4$. Moreover there is a locally bounded function $c(t)$ such that $|y'(t+h)-y'(t)| \leqslant c(t)\,h\log(1/h)$ for any real t and any positive h bounded by* 1.

Proof. From Lemma 9, differential quotients of y on $[0, 1] \cap D$ are uniformly bounded. Moreover, differences of differential quotients tend uniformly to zero when extreme evaluation points are close enough. The function y is uniformly continuous on $[0, 1]$, has a unique continuous extension. Differential quotients for the continuous extension give rise to the same inequalities as in Lemma 9. Because of this, y is differentiable on $[0, 1]$. If h is a positive number bounded by 1, if n is the integer such that $2^{-n-1} \leqslant h \leqslant 2^{-n}$, then $|y'(t+h)-y'(t)| \leqslant [2a+(n+1)\,b/2]\,2^{-n} \leqslant ch\log(1/h)$ provided c is properly chosen. To get a good bound for y', one can use Lemma 7. If t is in $[0, 1)$, we consider the sequence t_m, 2^{-m} times the integral part of $2^m t$ and we take for n the value 0. From Lemma 7

$$|y(1)-y(0)-[y(t_m+2^{-m})-y(t_m)]/2^{-m}| \leqslant a/2+b/4.$$

As m tends to ∞, the inequality becomes $|y(1)-y(0)-y'(t)| \leqslant a/2+b/4$. The argument can be modified to take care also of the case $t=1$. The proof is complete.

It is very difficult for $y(t)$ to be twice differentiable. If $y''(t)$ exists for any t of $(0, 1)$, then the function $v(t)$ as defined in the previous section vanishes on $[0, 1]$. This means that $v(0)=0$ and $v(1)=0$. This can happen if and only if he two finite differences of order 4 of the sequence $y(-2)$, $y(-1)$, $y(0)$, $y(1)$, $y(2)$, and $y(3)$ are equal to zero. When this is the case, then for t between 0 and 1, $y(t)=P(t)$ where P is the polynomial of interpolation through the points -2, -1, 0, 1, 2, and 3. Normally the degree of P should be 5, but here it does not exceed 3.

Remark. We come back to Theorem 5 for some statements. If $v(0)$ and $v(1)$ are not both zero, the function $v(t)$ is very wild inside $[0, 1]$: on any interval I, v take positive and negative values. This a consequence of formula (3). It follows from formula (4) that finite differences of order 2 inside I will not always be of the same sign. If $v(0)$ and $v(1)$ are not both zero, on any subinterval of $[0, 1]$, $y(t)$ cannot be convex, nor concave.

6. The Derivative of the Interpolant

We will now compute the derivative of y at any dyadic number. At first, we do the computation at the origin.

Lemma 11. *If $y(t)$ is the continuous extension of the interpolation of a sequence $\{y(n)\}_{n=-\infty}^{\infty}$, then*

$$y'(0)=\tfrac{4}{3}[y(1)-y(-1)]/2-\tfrac{1}{3}[y(2)-y(-2)]/4.$$

Proof. If $h=2^{-n}$, we set $p_n=[y(h)-y(-h)]/(2h)$. If $a=y(-4h)$, $b=y(-2h)$, $c=y(0)$, $d=y(2h)$, and $e=y(4h)$, then

$$\begin{aligned} y(-h)&=(-a+9b+9c-d)/16 \qquad \text{and}\\ y(h)&=(-b+9c+9d-e)/16,\\ p_n&=(a-10b+10d-e)/(32h)\\ &=\tfrac{5}{4}(d-b)/(4h)-\tfrac{1}{4}(e-a)/(8h)\\ &=\tfrac{5}{4}p_{n-1}-\tfrac{1}{4}p_{n-2} \qquad \text{if} \quad n>0. \end{aligned}$$

The solution of this recurrence is $p_n=c_1+c_2 4^{-n}$, where $c_1=\tfrac{4}{3}[y(1)-y(-1)]/2-\tfrac{1}{3}[y(2)-y(-2)]/4$. So $y'(0)$ is c_1.

THEOREM 12. If $n = 2^{-n}$ *and if t a dyadic number whose depth is smaller or equal to n, then for an interpolation function* $y(t)$,

$$y'(t) = \tfrac{4}{3}[y(t+h) - y(t-h)]/(2h) - \tfrac{1}{3}[y(t+2h) - y(t-2h)]/(4h).$$

Proof. Let $w(m)$ be the sequence $y(t+mh)$. The interpolation of this sequence is the function $w(u) = y(t+uh)$ as indicated by Theorem 2. If Lemma 11 is applied to the function w, we get the required formula for $y'(t)$.

7. Interpolation of an Exponential Sequence

If z is a complex number which is different from 0 and whose polar form is re^{ia}, we consider the sequence $z(n) = z^n$. Taking the real part and the imaginary part of $z(n)$, we get two real sequences, $x(n)$ and $y(n)$. The scheme (1) gives respective extensions $x(t)$ and $y(t)$ of these sequences. $z(t)$ is then the complex function $x(t) + iy(t)$. In this section, we carry through the harmonic analysis of the function $z(t)$.

THEOREM 11. *Let be* $z = re^{ia}$ $(r > 0)$ *and* $z_{j,n} = r^{1/2^j} e^{i(a+2\pi n)/2^j}$; *we introduce the rational function* $R(w) = (-1/w^3 + 9/w + 16 + 9w - w^3)/32$ *and the complex interpolation* $z(t)$ *of the exponential sequence* $z(n) = z^n$. *The following expansion is absolutely convergent*

$$z(t) = \sum_{n=-\infty}^{\infty} c_n r^t e^{i(a+2\pi n)t}, \qquad \textit{where} \quad c_n = \prod_{j=1}^{\infty} R(z_{j,n}).$$

Proof. The function $f(t) = z(t)/(r^t e^{iat})$ is a periodic function with period one. $f'(t)$ is well defined and has a modulus of continuity of order $h \log(1/h)$. This comes from Theorem 10. The Fourier series of f is then $\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}$. From the modulus of continuity of f' (see Zygmund [3, p. 71]), it can be said that the sequence $|c_n|$ is of the order of $\log(n)/n^2$.

At first, we will prove that c_0 is the infinite product of all the numbers $\{z_{n,0}\}_{n=1}^{\infty}$. Let us call by a_n the arithmetical mean of f of the following points of the real axis: $\{j/2^n\}$, $j = 0, 1,..., 2^n - 1$. It is known that a_n is the infinite sum of all c_k where k ranges over integral multiples of 2^n (see Lebesgue [2, p. 79]. The sequence a_n converge to c_0. a_0 is 1. We will show that for $n = 1, 2,..., a_n = R(z_{n,0})\, a_{n-1}$. By splitting the computation of a_n on even and odd multiples of $h = 2^{-n}$, we get that a_n is one half of a_{n-1} plus one half of the arithmetical mean of $f(t)$ over odd multiples of h which are in [0, 1]. If t is an odd multiple of h, we know that $z(t) = [-z(t-3h) + 9z(t-h) + 9z(t+h) - z(t+3h)]/16$.

So $f(t) = [-f(t-3h)/w^3 + 9f(t-h)/w + 9f(t-h)w - f(t+3h)w^3]/16$, where $w = r^h e^{iah}$. The arithmetical mean of f over odd multiples of h can be computed from the arithmetical mean of f over even multiples of h. $a_n = a_{n-1}/2 + a_{n-1}(-1/w^3 + 9/w + 9w + w^3)/32 = R(z_{n,0})\, a_{n-1}$. Then it follows that the infinite product of the numbers $R(z_{n,0})$ is convergent and is c_0.

To prove the similar fact that c_k is the infinite product of the $R(z_{n,k})$, it suffices to use another representation of the complex number z, $z = re^{i(a+2\pi k)}$. This completes the proof of the theorem.

THEOREM 12. *If $z = e^{ia}$ and if $z(t)$ is the interpolation of the sequence e^{ina}, then the Fourier coefficients of the function $z(t)\,e^{-iat}$ are nonnegative and their sum is equal to* 1. *For every t, $z(t)$ lies in the unit disk.*

Proof. Because z is complex number of modulus one, then all numbers $z_{j,n}$ are on the unit circle. If the modulus of $w = u + iv$ is one, then $R(w) = (2-u)(1+u)^2/4$. This number is a real number between 0 and 1. Each Fourier coefficient c_n is real and nonnegative. The sum of the coefficients is one, since this should be the value of the interpolation at $t = 0$, which value is one.

If m is a natural integer grater than 1, for each mth root ω_j of unity, $j = 1, 2, \ldots, m$, we consider the geometric sequence ω_j^n. The interpolation of this sequence will be a function $z_j(t)$. $z_j(t)$ is a periodic function of period m. Any interpolation according to the scheme (1) periodic of period m will be a linear combination of these m functions $\{z_j(t)\}_{j=1}^{m}$. Let us look at the

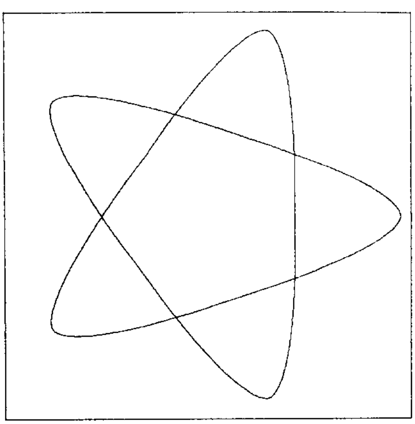

FIG. 3. Interpolation of a star-shaped pentagon.

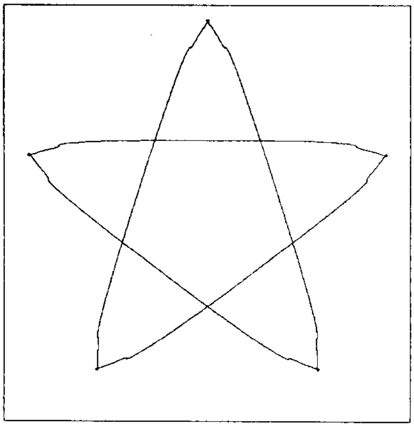

FIG. 4. Derivatives of the interpolation (times 0.4).

Fourier coefficients of $z_j(t)$ for the particular value $m = 5$. If ω_j are the complex numbers $e^{2\pi ij/5}$, then

$$z_j(t) = \sum_{m=-\infty}^{\infty} c_{j,n} e^{2\pi i(n+j/5)t}.$$

We remark that $z_5(t)$ is the constant function equal to 1, $z_4(t)$ and $z_3(t)$ are the respective conjugates to the functions $z_1(t)$ and $z_2(t)$.

Coefficients $c_{j,n}$

j \ n	−3	−2	−1	0	1	2	3
1	0,0006	0.0010	0.0180	0.9725	0.0062	0.0006	0.0003
2	0.0047	0.0131	0.2431	0.7045	0.0243	0.0045	0.0014

Within 3%, $z_1(t)$ is the circle $t \to 0.9725e^{2\pi it/5}$. Within 5%, $z_2(t)$ is the curve $0.7045e^{2\pi it/5} + 0.2431e^{-8\pi it/5}$. This happens in spite of the fact the concerned Fourier series are slowly convergent. In Figs. 3 and 4, we give the trace of $z_2(t)$ and of the derivative of $z_2(t)$.

8. Properties of the Fundamental Interpolation

In this section, we give various properties of the fundamental function $F(t)$. We will see that $|F|$ is bounded by one. F is positive on $(0, 1)$, negative on $(1, 2)$. On $[2, 3]$, $|F(t)|$ is quite small, less than 1/200. Figure 1 exhibits these properties, to prove them is harder.

LEMMA 13. *If t is between 2 and 3, then* $F(t) = -F(2t-3)/16$.

Proof. From Theorem 2, $F(t/2)$ is the interpolation of the sequence $F(n/2)$. According to Theorem 4, we get the functional relation: $F(t/2) = F(t) + \frac{9}{16}[F(t-1) + F(t+1)] - \frac{1}{16}[F(t-3) + (t+3)]$.

If t is between 4 and 6, since $F(t) = F(t-1) = F(t+1) = F(t+3) = 0$, then this relation becomes $F(t/2) = -F(t-3)/16$. Replacing t with $2t$, we get the expected relation.

THEOREM 14. *If t is between $3 - 2/2^n$ and $3 - 1/2^n$, where n is a positive integer, then the number $u = 2^n(t-3) + 3$ is between* 1 *and* 2 *and* $F(t) = (-1/16)^n F(u)$.

The evaluation of F on $[2, 3]$ comes from evaluation of F on $[1, 2]$.

LEMMA 15. $F(t) = (3-t)(2-t)(1-t)/6 - 4F(1+t) + F(4-t)$ *when* $t \in [1, 2]$.

Proof. The interpolation of the sequence $(3-n)(2-n)(1-n)/6$ is the function $y(t) = (3-t)(2-t)(1-t)/6$. From Theorem 4, $y(t) = \sum_{k=-1}^{4} y(k)\, F(t-k)$, if $t \in [1, 2]$. Since $y(-1) = 4$, $y(0) = 1$, $y(1) = y(2) = y(3) = 0$, $y(4) = -1$, and $F(t-4) = F(4-t)$, the formula is proven.

THEOREM 16. *The function F is negative on* $(1, 2)$ *and is bounded by* 0.13 *in absolute value.*

Proof. Let us introduce the cubic polynomial $p(t) = [(3-t)(2-t)(1-t)]/6$ and the function $H(t) = F(t)/p(t)$. As F is differentiable, H is well defined on $(1, 2)$ and is bounded. Lemma 15 and Theorem 14 can be used to show that H satisfies the functional equation:

$$H(t) = 1 + a(t)\, H(f(t)) + b(t)\, H(g(t)),$$

where f, g, a, and b are defined as follows:

If $n \geqslant 2$ and if $1 + 1/2^n \leqslant t \leqslant 1 + 2/2^n$, then

$$f(t) = 3 - (t-1)\, 2^n, \qquad g(t) = 2t - 1,$$

$$a(t) = p[f(t)]/[(-16)^n\, p(t)] \qquad \text{and} \qquad b(t) = p[g(t)]/[4p(t)].$$

If $n \geqslant 2$ and if $2 - 2/2^n \leqslant t \leqslant 2 - 1/2^n$, then

$$f(t) = 3 - (2-t)\, 2^n, \qquad g(t) = 5 - 2t,$$

$$a(t) = -4p[f(t)]/[(-16)^n\, p(t)] \qquad \text{and}$$

$$b(t) = -p[g(t)]/[16p(t)].$$

We now look for bounds for $a(t)$ and $b(t)$. If $n \geqslant 2$ and if $1+1/2^n \leqslant t \leqslant 1+2/2^n$, then we set $u=f(t)=3-(t-1)\,2^n$.

$$a(t)=\frac{(3-u)(2-u)(1-u)}{[(-16)^n(3-t)(2-t)(1-t)]}$$
$$=\frac{(2-u)(u-1)}{[(-8)^n(3-t)(2-t)]}.$$

Since $t \in [1, 3/2]$, $(3-t)(2-t) \geqslant 3/4$. Since $u \in [1, 2]$, $(2-u)(u-1) \leqslant \frac{1}{4}$. From these inequalities and from the fact that $n \geqslant 2$, it follows that $|a(t)| \leqslant \frac{1}{192}$.

If $n \geqslant 2$ and if $2-2/2^n \leqslant t \leqslant 2-1/2^n$, then we set $u=f(t)=3-(2-t)\,2^n$,

$$a(t)=-\frac{4(3-u)(2-u)(1-u)}{[(-16)^n(3-t)(2-t)(1-t)]}$$
$$=-\frac{4(2-u)(1-u)}{[(-8)^n(3-t)(1-t)]}.$$

Since $t \in [3/2, 2]$, $(3-t)(t-1) \geqslant 3/4$. Since $u \in [1, 2]$, $(2-u)(1-u) \leqslant \frac{1}{4}$. From these inequalities and from the fact that $n \geqslant 2$, it follows that $|a(t)| \leqslant \frac{1}{48}$.

If $t \in [1, \frac{3}{2}]$, we set $u=g(t)=2t-1$,

$$b(t)=\frac{(3-u)(2-u)(1-u)}{4(3-t)(2-t)(1-t)}=(2-u)/(3-t)=(3-2t)/(3-t),$$

$|b(t)| \leqslant \frac{1}{2}$. It will be important to note that at this point $b(t)$ is nonnegative.

If $t \in [3/2, 2]$, we set $u=g(t)=5-2t$,

$$b(t)=-\frac{(3-u)(2-u)(1-u)}{16(3-t)(2-t)(1-t)}=-(2-u)/[4(3-t)]$$
$$=-(2t-3)/[4(3-t)],$$

$$|b(t)| \leqslant \tfrac{1}{4}.$$

The inequalities we need for $a(t)$ and $b(t)$ are completed. Let us study the function $H(t)$. Let be M the supremum of $|H(t)|$ and m the minimum of $H(t)$ on $(1, 2)$. We would like to prove that $m>0$. At first, we remark that M satisfies the inequality: $M \leqslant 1+(|a(t)|+|b(t)|)\,M$. If $k=1/2+1/192$, then $M \leqslant 1/(1-k)$. When t belongs to $[1/2, 2)$, then we minorize H,

$$H(t) \geqslant 1-(|a(t)|+|b(t)|)\,M \geqslant 1-(1/48+1/4)/(1-k) \geqslant 0.$$

Let us assume for a moment that $m \leqslant 0$. If $t \in (1, \frac{3}{2})$, then $H(t) \geqslant 1 - |a(t)|\, M + b(t)\, m$. We have the inequality $m \geqslant 1 - M/48 + m/2$. Such an inequality cannot be true if m is not positive.

If we come back to the function F, we have $|F(t)| \leqslant M(3-t)(2-t)(t-1)/6$ if $t \in [1, 2]$. The maximum value of the function $(3-t)(2-t)(t-1)/6$ is 0.06415. $|F(t)| \leqslant 0.13$ whenever $t \in [1, 2]$. The proof is complete.

Remark. The bound for the minimal value of F on $[1, 2]$ is somewhat crude. Numerical computations suggest that -0.074 is a lower bound for $F(t)$ on $[1, 2]$.

THEOREM 17. *The function $|F|$ is bounded by one.*

Proof. We introduce six sequences of complex numbers $\{z_j(n)\}_{j=1}^{6}: z_j(n) = e^{\pi ijn/3}$. $z_j(t)$ is the interpolation of the corresponding sequence. If $A(t)$ is the arithmetical mean of these six functions, then $A(0) = 1$, $A(1) = A(2) = A(3) = A(4) = A(5) = 0$. If t is between 0 and 3, then $A(t) = F(t)$ as shown by Theorem 4. By Theorem 12, each value $z_j(t)$ lies in unit-disk, the same is true for each value of $A(t) = F(t)$. $|F(t)| \leqslant 1$.

LEMMA 18. $F(t) = (3-t)(2-t)(1-t)/6 - 4F(1+t) + 10F(2+t)$ *when* $t \in [0, 1]$.

Proof. The interpolation of the sequence $(3-n)(2-n)(1-n)/6$ is the function $y(t) = (3-t)(2-t)(1-t)/6$. From Theorem 4, if t is in $[0, 1]$, then $y(t) = \sum_{k=-2}^{3} y(k)\, F(t-k)$. $y(-2) = 10$, $y(-1) = 4$, $y(0) = 1$, $y(1) = y(2) = y(3) = 0$, the formula is proven.

THEOREM 19. *The function F is positive on $(-1, 1)$.*

Proof. F is an even function, we consider a value $t \in [0, 1)$. By Lemma 18, $F(t) = (3-t)(2-t)(1-t)/6 - 4F(1+t) + 10F(2+t)$. $-4F(1+t)$ is positive (Theorem 16). It suffices to prove that $10\, |F(2+t)| \leqslant (3-t)(2-t)(1-t)/6$. We will use Theorem 14. Let n be the positive integer such that t is between $1 - 2/2^n$ and $1 - 1/2^n$, then the number $u = 2^n(t-1) + 3$ is between 1 and 2 and $F(t+2) = (-1/16)^n\, F(u)$. $|F(t+2)| \leqslant 0.13/16^n$ according to Theorem 16. We know that $1/2^n \leqslant (1-t)$. So $|F(t+2)| \leqslant 0.13(1-t)/8$ and $10\, |F(2+t) \leqslant (3-t)(2-t)(1-t)/6$.

9. ERROR IN INTERPOLATION

Let be $f(t)$ a real function defined on the real axis. We use the following sequence: $y(n) = f(n)$ for relative integers n. $y(t)$ is the interpolation of the sequence $y(n)$ according to the scheme (1). The error in the interpolation is the function $e(t) = f(t) - y(t)$. We should like to get simple bounds for $e(t)$.

THEOREM 20. *Let be $y(t)$ the interpolating function for a function f. If $t \in [0, 1]$ and if $p(t)$ is the cubic polynomial such that $p(n) = f(n)$, $n = -1, 0,$ 1 and 2, then*

$$|f(t) - y(t)| \leqslant |f(t) - p(t)| + (|f(3) - p(3)| + |f(-2) - p(-2)|)/200.$$

Proof. We use the triangular inequality.

$$\begin{aligned} |f(t) - y(t))| &= |f(t) - p(t) + p(t) - y(t)| \\ &\leqslant |f(t) - p(t)| + |p(t) - y(t)|. \end{aligned}$$

If $t \in [0, 1]$, then Theorem 4 tells us that

$$\begin{aligned} p(t) - y(t) &= \sum_{k=-2}^{3} [p(k) - f(k)]\, F(t-k) \\ &= [p(-2) - f(-2)]\, F(t+2) + [p(3) - f(3)]\, F(t-3), \end{aligned}$$

$|F(t+2)|$ and $|F(t-3)|$ are bounded by $\frac{1}{200}$. $|p(t) - y(t)| \leqslant (|p(-2) - f(-2)| + |p(3) - f(3)|)/200$.

We close this section with an example. We take as function $f(t)$ the fourth power of t.

THEOREM 21. *If $y(t)$ is the interpolation according to the scheme* (1) *of the sequence n^4, then*

$$y(t/2) = y(t)/16 - (9/16) \sum_{n=-\infty}^{\infty} F(t - 2n - 1).$$

Proof. If $w(n) = y(n/2) - y(n)/16$ and if $w(t)$ is the interpolation according to the scheme (1) of the sequence $w(n)$, then $w(t) = y(t/2) - y(t)/16$. If n is even, $w(n) = 0$. If n is odd,

$$\begin{aligned} y(n/2) &= [-(n/2 - 3/2)^4 + 9(n/2 - 1/2)^4 + 9(n/2 + 1/2)^4 \\ &\quad - (n/2 + 3/2)^4]/16 \\ &= (n^4 - 9)/16 \end{aligned}$$

and then $w(n) = -9/16$. From Theorem 4, it follows that

$$w(t) = -\tfrac{9}{16} \sum_{n=-\infty}^{\infty} F(t-2n-1).$$

THEOREM 22. *If $y(t)$ is the interpolation of the sequence n^4 and $e(t) = t^4 - y(t)$, then $e(t)$ is a periodic function of period one bounded below by* 0 *and bounded above by* 3/5.

Proof. Let $q(t) = (t+1)^4 - t^4$, q is a cubic polynomial. The interpolation of the sequence $[(n+1)^4 - n^4] = q(n)$ is $[y(t+1) - y(t)] = q(t)$. It follows that

$$\begin{aligned} e(t+1) - e(t) &= (t+1)^4 - y(t+1) - [t^4 - y(t)] \\ &= q(t) - q(t) = 0. \end{aligned}$$

This proves that e is periodic of period one. Let be $z_2(t)$ the interpolation according to scheme (1) of the sequence $(-1)^n$, the function $\sum_{n=-\infty}^{\infty} F(t-2n-1)$ is equal to $[1 - z_2(t)]/2$. The previous theorem says that $y(t/2) = y(t)/16 - (9/32)[1 - z_2(t)]$. So $e(t) = e(2t)/16 + (9/32)[1 - z_2(2t)]$. If t is a dyadic number of depth n,

$$e(t) = (9/32) \sum_{k=0}^{n-1} [1 - z_2(2^{k+1}t)]/16^k.$$

This is true because the function e vanishes at any integer.

According to Theorem 12, $|z_2(t)| \leqslant 1$. The following inequality holds for the function $e(t)$: $0 \leqslant e(t) \leqslant \frac{3}{5}$. This completes the proof.

If a numerical computation is done, the maximal value of $e(t)$ is found to be about 0.57. The bound $\frac{3}{5}$ of Theorem 20 is quite close.

10. INTERPOLATION IN A FINITE INTERVAL

If $\{y_n\}_{n=0}^{m}$ is a sequence of real values, we would like to find a function $y(t)$ such that $y(n) = y_n$ for integers between 0 and m. We assume that m is not less than 3, we introduce two polynomials $p(t)$ and $q(t)$: p is the cubic polynomial such that $p(0) = y_0$, $p(1) = y_1$, $p(2) = y_2$, and $p(3) = y_3$ and q is the cubic polynomial such that $q(m) = y_m$, $q(m-1) = y_{m-1}$, $q(m-2) = y_{m-2}$, and $q(m-3) = y_{m-3}$. p will be called the left cubic polynomial and q the right one. We define the sequence $y(n)$ as $p(n)$ if $n < 0$, y_n if $0 \leqslant n \leqslant m$ and $q(n)$ if $n > m$. $y(t)$ will then be the interpolation of the sequence $y(n)$ according to the scheme (1.) We will say that $y(t)$ is the interpolation of the sequence $\{y_n\}_{n=0}^{m}$ according to the scheme (1e). The two next lemmas will help us to compute $y(t)$ when t belongs to $[0, m]$.

LEMMA 23. *If $y(t)$ is the interpolation of the sequence $\{y_n\}_{n=0}^{m}$ according to the scheme* (1e), *if $p(t)$ is the left cubic polynomial and if $t \in [0, 1]$, then $y(t) = p(t)$.*

Proof. We use Theorem 4. $y(t) = \sum_{k=-2}^{3} y(k)\, F(t-k)$, $y(t) = \sum_{k=-2}^{3} p(k)\, F(t-k) = p(t)$.

LEMMA 24. *If $y(t)$ is the interpolation of the sequence $\{y_n\}_{n=0}^{m}$ according to the scheme* (1e) *and if h is nonnegative integral power of $\frac{1}{2}$, then $y(h/2) = [5y(0) + 15y(h) - 5y(2h) + y(3h)]/16$.*

Proof. We use the following Lagrange polynomials. $L_0(t) = (1-t)(2-t)(3-t)/6$, $L_1(t) = t(2-t)(3-t)/2$, $L_2(t) = t(t-1)(3-t)/2$, $L_3(t) = t(t-1)(t-2)/6$. If f is a cubic polynomial, $f(1/2) = \sum_{k=0}^{3} f(k)\, L_k(1/2) = [5f(0) + 15f(1) - 5f(2) + f(3)]/16$. If h is nonnegative integral power of $\frac{1}{2}$, then the values $y(0)$, $y(h)$, $y(2h)$, and $y(3h)$ are, respectively, $p(0)$, $p(h)$, $p(2h)$, and $p(3h)$ as it can be seen. When h is smaller or equal than $\frac{1}{4}$, this comes from Lemma 22. When $h = \frac{1}{2}$, $y(\frac{1}{2}) = p(\frac{1}{2})$ according to Lemma 23, $y(\frac{3}{2}) = [-y(0) + 9y(1) + 9y(2) - y(3)]/16 = [-p(0) + 9p(1) + 9p(2) - p(3)]/16 = p(3/2)$. All other cases come from the fact that $y(k) = p(k)$, $k = 0, 1, 2, 3$.

It follows that $y(h/2) = p(h/2) = [5p(0) + 15p(h) - 5p(2h) + p(3h)]/16 = [5y(0) + 15y(h) - 5y(2h) + y(3h)]/16$. This completes the proof.

If n is a nonnegative integer, if h is 2^{-n}, then

$$y(h/2) = [5y(0) + 15y(h) - 5y(2h) + y(3h)]/16,$$

$$y(m - h/2) = [5y(m) + 15y(m-h) - 5y(m-2h) + y(m-3h)]/16.$$

For any other dyadic number t of depth $n+1$ in $[0, m]$,

$$y(t) = [-y(t-3h) + 9y(t-h) + 9y(t+h) - y(t+3h)]/16.$$

With these formulas, it is possible to compute the function $y(t)$ on $[0, m]$ without explicit use of the cubics $p(t)$ and $q(t)$.

11. A GRAPHICAL APPLICATION

The interpolation scheme can be useful for the generation of closed or open curves in the plane. We begin with closed curves. If $\{(x_i, y_i)\}_{i=1}^{m}$ are m given points of the plane, we may like to find a closed curve $x(t)$, $y(t)$, where $t \in [0, m]$, $x(i) = x_i$, $y(i) = y_i$, $i = 0, 1, ..., m$. By assumption, $x_0 = x_m$ and $y_0 = y_m$. The interpolation can be done as follows. If $n = km + i$ with $1 \leqslant i \leqslant m$, then $x(n) = x_i$ and $y(n) = y_i$. One extends the function $x(t)$ and $y(t)$ to the real axis according to the scheme (1) and then by continuous extension.

$$x(t) = \sum_{k=1}^{m} x_k F_m(t-k), \qquad y(t) = \sum_{k=1}^{m} y_k F_m(t-k),$$

where $F_m(t)$ is the periodic function $\sum_{j=-\infty}^{\infty} F(t+jm)$. This last series is in fact a finite sum.

If it is rather an open curve we want to draw through $m+1$ points $\{(x_i, y_i)\}_{i=0}^{m}$, we use $x(t)$ and $y(t)$, the interpolations according to the scheme (1e) with respect to each sequence $\{x_i\}$ and $\{y_i\}$ as described in Section 10.

In both situations, the curve $x(t)$, $y(t)$ is a continuously differentiable curve. It is almost twice differentiable. If one point like x_3, y_3 is moved while others are fixed, the curve $x(t)$, $y(t)$ will not change for values of t in $[6, m]$, the change will be very small for $t \in [0, 1]$ and rather small for $t \in [1, 2]$. The main changes will happen for t in $[2, 4]$. A curve fitting with this property was called by Akima [1] a local procedure.

We conclude by some open problems about the fundamental function $F(t)$. Is it true that F is decreasing on $[0, 1]$? According to Figs. 1 and 2, this seems the case, but how can it be proven? Is it true that F has a unique minimum on $[1, 2]$? Is it possible to find exactly the zeroes of the function $F'(t)$? Is it true that $F''(t)$ does not exist at any point t of $(-3, 3)$?

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