Computer Aided Geometric Design, USA, 14, 1997, p. 1-14.

Asymptotically Optimal Disposition of Tangent Points for Approximation of Smooth Convex Surfaces by Polygonal Functions

A.A.Ligun[1], A.A.Shumeiko[1], S.P.Radzevitch[1], E.D.Goodman[2]
[1] Dneprodzerzhinsk State Technical University, Dneprodzerzhinsk, Ukraine
[2] Case Center for Computer-Aided Engineering and Manufacturing, Michigan State University, East Lansing, MI 48824 USA.

Keywords: approximation, surface polygonization, exterior tangent facets

Running Title: Asymptotically Optimal Approximation of Surfaces by Polygonal Functions

Abstract

Many applications require that smooth convex surfaces be approximated by polygonal functions. While there exist methods for determining optimal tangent points for definition of exterior facetings of such surfaces, they are not practical to compute for many real-world problems. Asymptotically optimal tangent points are nearly as useful, and using the result presented here, are far more easily computed.

A ubiquitous problem in engineering geometry is the description of a part surface by means of a mathematical form appropriate for the uses to be made of it. In this article, we investigate one such problem – namely that of finding the best description (in an asymptotic sense) for a smooth convex surface in terms of a set of tangent planes (exterior facets). While there exist methods for determining optimal tangent points for definition of exterior facetings of such surfaces, they are not practical to compute for many real-world problems. Asymptotically optimal tangent points are nearly as useful, and using the result presented here, are far more easily computed ([1], [5]).

Before the result is derived, it is necessary to define the sense in which it will be asserted to be an asymptotically optimal description of the surface. To that end, some necessary concepts are introduced.

Given a convex smooth surface defined by the equation

$$z = z(x, y) \quad ((x, y) \in D), \tag{1}$$

we'll say that domain D is partitioned into $D_1, D_2, \ldots, D_n$, if

$$\bigcup_{i=1}^{n} D_i = D, \tag{2}$$

and

$$\text{mes}\,(D_i \cap D_j) = 0 \;\; (i \neq j), \tag{3}$$

(here mes (D) is a measure on domain D).

By a polygonal function (the two-dimensional analog of piecewise linear curve) we'll mean a continuous function consisting of "pieces"(patches) of planes, or facets.

An exact definition of a polygonal function follows:

A function $f_n(x, y)$ defined on domain D is called *polygonal* if it is continuous on the set D and there exists a partition of domain D into parts $D_1, D_2, \ldots, D_n$ such that for every i there exists a plane P_i such that all points $(x, y, f_n(x, y))$ $((x, y) \in D)$ belong to the same plane P_i.

The polygonal description of surfaces has wide application in many fields – note, for example, that in machine-building, such a problem appears in the forming of a part by a plane cutter, grinding disk, etc. The problem of optimal machining of a surface and many other problems are reducible to the problem of polygonal approximation of the surface.

Under the approximation of convex surface by polygonal functions we can consider two problems – descriptions by an external (consisting of tangent planes) polygonal function (Fig. 1) and by an internal (consisting of interpolational planes) polygonal function (Fig. 2).

The main problem is the determination of an optimal disposition of tangent points (in the first case) or interpolation points (in the second case).

Now we formulate a definition for the quality of approximation of a surface and an asymptotically optimal disposition of nodes (the tangent points or interpolation points).

For clarity of exposition, we'll suppose that the part surface is given in the form of (1) and consider the problem of external milling.

Denote this surface by Θ. For any point $M(x, y, z)$, let

$$B_\varepsilon(M) = B_\varepsilon(x, y, z)$$

denote a ball with radius ε and cutter position described at point M.

The union $T_\varepsilon(\Theta)$ of all balls with radius ε and with centers at points on surface Θ will be called the ε-layer of surface Θ.

The set of points $M(x, y, z)$ $((x, y) \in D)$ such that $M \in T_\varepsilon(\Theta)$ and $z \geq z(x, y)$ will be called the external ε-layer of surface Θ and will be denoted by $T_\varepsilon^+(\Theta)$, and those such that $z \leq z(x, y)$ will be called the internal ε-layer and will be denoted by $T_\varepsilon^-(\Theta)$. We'll say that polygonal function $z_n(x, y)$ is ε- *permissible* for surface Θ if

$$z_n(x, y) \subset T_\varepsilon^+(\Theta) \qquad \forall (x, y) \in D \tag{4}$$

(i.e. function $z_n(x, y)$ belongs to $T_\varepsilon^+(\Theta)$).

The polygonal function $z_{n_{opt}}(x, y)$ (where integer $n_{opt} = n_{opt}(\varepsilon)$) will be called ε-*optimal* for surface Θ if it is ε-permissible and for any ε- permissible function $z_n(x, y)$ the inequality $n \geq n_{opt}$ is valid. Therefore the ε- optimal polygonal function has the minimal number of patches among all ε-permissible functions for surface Θ.

Now let $\varepsilon \to 0$. The sequence of ε-permissible polygonal functions $\{z_{n_*}(x, y)\}$ (where integer $n_* = n_*(\varepsilon)$) will be called *asymptotically optimal* if

$$\lim_{\varepsilon \to 0} \frac{n_*(\varepsilon)}{n_{opt}(\varepsilon)} = 1. \tag{5}$$

This relationship is denotes as

$$n_* = n_{opt}(1 + o(1)). \tag{6}$$

Let $N_i = (x_i, y_i)$ be arbitrary points of domain D. The equation of the plane tangent to surface Θ at point N_i is

$$(7)\quad P_i(x,y,z) = \frac{\partial z(x,y)}{\partial x}\Big|_{(x_i,y_i)}(x-x_i)+\frac{\partial z(x,y)}{\partial y}\Big|_{(x_i,y_i)}(y-y_i)-(z-z(x_i,y_i)) = 0.$$

If surface Θ is convex, then the lower tangent surface to all surfaces P_i is a polygonal function.

Our problem is to find a distribution of points N_i such that this bounding surface is in the ε-layer $T^{+}_{\varepsilon^*}(\Theta)$, where

$$\lim_{n\to\infty} \frac{\varepsilon^*}{\varepsilon} = 1, \tag{8}$$

or, similarly

$$\varepsilon^* = \varepsilon(1 + o(1)). \tag{9}$$

An exact optimal solution to such problems reduces to a difficult minimax problem and can be found only in easy case – for instance, if the surface is a sphere, a paraboloid of revolution or a parabolic cylinder. For real-world problems, when surfaces are composite and the number of tangent points (or interpolation points) is sufficiently large, the exact solution of such problems is practically impossible. In such a case, with a requirement for a high-precision description of the surface by a polygonal function, asymptotically optimal methods are almost as effective as the optimal methods. Then the realization of asymptotically optimal methods via computer is relatively simple and convenient. That is a key justification for using asymptotic methods in the solution of engineering problems.

This problem may be decomposed into three sub-problems:
The first is to build a constructive algorithm for finding the tangent points, the second is to show that the lower bounding surface of the tangent planes at these points lies in the ε-layer, and the third is to show that any other polygonal ε-permissible function for the surface consists of no fewer planes, i.e., that there is no other polygonal function which can yield a better result.

For solution of engineering problems, it is enough to solve the first of the problems stated. The second problem must be solved to prove the assumption of the first problem. The third (which is the most difficult from a mathematical viewpoint) is necessary only for user confidence in the fact that the given algorithm could be simplified, or adapted to the user's problems, but it cannot be improved upon in the sense of ε-optimality. The results obtained in this article are essentially based on the results obtained in references [3] and [4], and are their one-dimensional analogs.

Let $\Gamma(t) = (x(t), y(t), z(t))$ be a parametrically defined curve and

$$\Delta_m = \{0 = t_0 < t_1 < ... < t_m = T\} \tag{10}$$

be an arbitrary partition of the domain of the parameter.

Let $\gamma(t, \Delta_m)$ be a function that coincides over each interval $[t_i, t_{i+1}]$ $(i = 0, 1, ..., m-1)$ with the tangent line to Γ at the some point $\tilde{t_i} \in (t_i, t_{t+1})$. If $\gamma(t, \Delta_m) \subset T_\varepsilon(\Gamma)$, then $\gamma(t, \Delta_m)$ will be called ε-permissible. The function $\gamma^*(t, \Delta_{m_o})$ (where integer $m_o = m_o(\varepsilon)$) will be called ε-optimal for curve Γ if it is ε- permissible and for any ε-permissible function $\gamma(t, \Delta_m)$ the inequality $m \geq m_o(\varepsilon)$ is valid.

The sequence of ε-permissible functions $\{\gamma^*(t,\Delta_{m_*})\}$ (where integer $m_* = m_*(\varepsilon)$) will be called asymptotically optimal if

$$\lim_{\varepsilon\to 0}\frac{m_*(\varepsilon)}{n_o(\varepsilon)} = 1. \tag{11}$$

For parametric function $\Gamma(t)= (x(t), y(t), z(t))$ such that x, y, z are twice continuously differentiable on the interval $[0, T]$ we shall introduce the following function:

$$\Phi(\Gamma,t) = \Phi(t) = \sqrt{\frac{|\overline{\Gamma}'(t)\times\overline{\Gamma}''(t)|}{|\overline{\Gamma}'(t)|}}, \tag{12}$$

where $\overline{a}\times\overline{b}$ is the vector product of vectors $\overline{a}$ and $\overline{b}$, and $|\overline{a}|$ is the length of vector $\overline{a}$.

In coordinate form, the function Φ can be written as:

$$\Phi(t) = \sqrt[4]{\frac{(x'y''-x''y')^2+(y'z''-y''z')^2+(z'x''-z''x')^2}{(x')^2+(y')^2+(z')^2}} \tag{13}$$

Theorem A.*Let functions $x(t), y(t)$ and $z(t)$ determining the curve $\Gamma(t)$ be three times continuously differentiable on [0,T] and let $\alpha\in(0,1/2)$. Then for any $\varepsilon>0$, the number of asymptotically optimal nodes of function $\{\gamma^*(t,\Delta_m)\}$ is equal to*

$$m = m(\varepsilon) = \left[\frac{1}{\sqrt{\varepsilon}}\int_0^T \Phi(t)dt\right]+1, \tag{14}$$

where [·] denotes the integer part and where nodes $\Delta_m = \{t_i\}_{i=0}^m$ are defined from equations

$$\int_0^{t_i}(\Phi(t)+m^{-\alpha})dt = \frac{i}{m}\int_0^T(\Phi(t)+m^{-\alpha})dt \ \ (i=0,1,2,...,m). \tag{15}$$

Then the piecewise linear curve passing through tangent points $\Gamma_{i+1/2}$:

$$\gamma^*(t,\Delta_m) = (x'_{i+1/2}(t-t_{i+1/2})+x_{i+1/2};$$

$$y'_{i+1/2}(t-t_{i+1/2})+y_{i+1/2}; z'_{i+1/2}(t-t_{i+1/2})+z_{i+1/2})$$

will be asymptotically optimal for curve Γ (when $\varepsilon\to 0$), where we denote $t_{i+1/2} = (t_{i+1}+t_i)/2$ and

$$x_{i+1/2} = x(t_{i+1/2}),\ \ x'_{i+1/2} = x'(t_{i+1/2})$$

and the some is true for $y(t)$ and $z(t)$.

The algorithm for asymptotically optimal choice of nodes (15) is illustrated in Fig. 3.

Before we begin to describe the algorithm for building the optimal polygonal function, we'll develop some auxiliary constructions which we will require. At each point $(x_{i,j}, y_{i,j}, z(x_{i,j}, y_{i,j}))$, the two (orthogonal) principal directions of curvature are defined, which at our point lie along tangent lines to the curves defined by the Cauchy problem:

$$\begin{cases}(EB-FA)(dx)^2+(EC-GA)dxdy+(FC-GB)(dy)^2=0;\\ y(x_{i,j}) = y_{i,j}\end{cases}$$

where

$$A = \frac{r}{\sqrt{1+p^2+q^2}};\;\; B = \frac{s}{\sqrt{1+p^2+q^2}};\;\; C = \frac{l}{\sqrt{1+p^2+q^2}};$$

$$F = 1+p^2;\;\; E = pq;\;\; G = 1+q^2;$$

and

$$p = \frac{\partial z(x,y)}{\partial x};\; q = \frac{\partial z(x,y)}{\partial y};\; r = \frac{\partial^2 z(x,y)}{\partial^2 x};\; l = \frac{\partial^2 z(x,y)}{\partial^2 x};\; s = \frac{\partial^2 z(x,y)}{\partial x \partial y}.$$

There are two equations of first order:

$$\begin{cases} y' = f_i(x,y); \\ y(x_{i,j}) = y_{i,j} \end{cases} \tag{16}$$

or, equivalently,

$$\begin{cases} x' = g_i(x,y); \\ x(y_{i,j}) = x_{i,j} \end{cases} \tag{17}$$

where for $i = 1,2$

$$f_i = \frac{1}{2(FC-GB)}\left(GA - EC + (-1)^i\sqrt{(EC-AG)^2 - 4(FC-GB)(EB-FA)}\right),$$

and, similarly,

$$g_i = \frac{1}{2(EB-FA)}\left(GA - EC + (-1)^i\sqrt{(EC-AG)^2 - 4(FC-GB)(EB-FA)}\right).$$

In the same way, through every point $(x_{i,j}, y_{i,j}, z(x_{i,j}, y_{i,j}))$ there will pass two orthogonal curves, at every point of which the tangent lines are the principal directions of curvature. Denote them by

$$\begin{cases} x = t; \\ y = \varphi(t); \\ z = z(t, \varphi(t)); \end{cases} \tag{18}$$

and

$$\begin{cases} x = \varphi_{i,j}(t); \\ y = t; \\ z = z(\varphi_{i,j}(t), t). \end{cases} \tag{19}$$

Although we noted that point $(x_{0,0}, y_{0,0}, z(x_{0,0}, y_{0,0}))$ was arbitrary, it is better to choose the first point such that $(x_{0,0}, y_{0,0})$ is near the center of gravity for domain D. Let

$$\begin{cases} x = t; \\ y = \varphi_{0,0}(t); \\ z = z(t, \varphi_{0,0}(t)); \end{cases} \tag{20}$$

and

$$\begin{cases} x = \phi_{0,0}(t); \\ y = t; \\ z = z(\phi_{0,0}(t), t). \end{cases} \tag{21}$$

Now, thanks to the algorithm of Theorem A, we construct the collection of points t_i using curve $\Gamma(t)$, with error 2ε, i.e., we define the number

$$n = \left[\frac{1}{4\sqrt{\varepsilon}} \int_0^T \Phi(\Gamma, t)dt \right] + 1, \tag{22}$$

and the nodes from the condition:

$$\int_0^{t_i} (\Phi(\Gamma, t) + n^{-\alpha})dt = \frac{i}{n} \int_0^T (\Phi(\Gamma, t) + n^{-\alpha})dt \ \ (i = 0, 1, 2, ..., n). \tag{23}$$

And from curve $\Gamma_0(t)$ we define the collection of points $t_{j,0}$ with error $\varepsilon\ \sqrt{3}/2$; the number of partitions will be:

$$n_0 = \left[\frac{1}{2\sqrt[4]{3\varepsilon^2}} \int_0^{T_0} \Phi(\Gamma_0, t)dt \right] + 1, \tag{24}$$

and the nodes, from the condition, are:

$$\int_0^{t_{j,0}} (\Phi(\Gamma_0, t) + n_0^{-\alpha})dt = \frac{j}{n_0} \int_0^{T_0} (\Phi(\Gamma_0, t) + n_0^{-\alpha})dt \ \ (i = 0, 1, 2, ..., n_0). \tag{25}$$

Then, in both cases, we shall choose points corresponding to the values of parameter t_i $(i = 0, 1, ..., n)$ and $t_{j,0}$ $(j = 0, 1, ..., n_0)$, that lie in domain D (Fig. 4). By $\Gamma_i(t)$ we shall denote the direction of one of the principal curvatures and passing through point $\Gamma_i = \Gamma(t_i)$ orthogonally to $\Gamma(t)$ (Fig. 5), and on each of the given curves define the set $t_{j,i}$ with error $\varepsilon\ \sqrt{3}/2$ (from Theorem A). Supplement the set of points obtained,

$$\Gamma_{i,j} = (x_{i,j}, y_{i,j}, z(x_{i,j}, y_{i,j})) \tag{26}$$

by points

$$\Gamma_{i+1/2,j+1/2} = (x_{i+1/2,j+1/2}, y_{i+1/2,j+1/2}, z(x_{i+1/2,j+1/2}, y_{i+1/2,j+1/2})) \tag{27}$$

where

$$x_{i+1/2,j+1/2} = \frac{1}{4}(x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1}), \tag{28}$$

$$y_{i+1/2,j+1/2} = \frac{1}{4}(y_{i,j} + y_{i+1,j} + y_{i,j+1} + y_{i+1,j+1}). \tag{29}$$

Then the polygonal surface formed from the tangent planes to $z = z(x, y)$ at the points $\Gamma_{i,j}$ and $\Gamma_{i+1/2,j+1/2}$, i.e. (Fig. 6 - 7)

$$\frac{\partial z(x,y)}{\partial x}\Big|_{\Gamma_{i,j}} (x - x_{i,j}) + \frac{\partial z(x,y)}{\partial y}\Big|_{\Gamma_{i,j}} (y - y_{i,j}) - (z - z(x_{i,j}, y_{i,j})) = 0 \tag{30}$$

and

$$\frac{\partial z(x,y)}{\partial x}\Big|_{\Gamma_{i+1/2,j+1/2}} (x - x_{i+1/2,j+1/2}) + \frac{\partial z(x,y)}{\partial y}\Big|_{\Gamma_{i+1/2,j+1/2}} (y - y_{i+1/2,j+1/2}) - \tag{31}$$

$$\big(z - z(x_{i+1/2,j+1/2}, y_{i+1/2,j+1/2})\big) = 0$$

will be asymptotically optimal from above. To prove this fact, we first show that the given polygonal function lies within the ε-layer of surface Θ. As regards the possibility that it lies in the upper part of the ε-layer, then it would follow from the condition of convexity upward of surface Θ and the construction method of the polygonal function (by the definition of convexity, that the surface will be convex upward on the domain D, if for any point $(x, y) \in D$ the corresponding point on any tangent plane lies higher than the corresponding point on the surface). Let us take point $x_{i,j} = 0$, $y_{i,j} = 0$. Then the tangent plane to the surface $z = z(x, y)$ at the point $(0, 0, z_{i,j})$ (here $z_{i,j} = z(0, 0)$) will have equation (see (7))

$$\frac{\partial z(x,y)}{\partial x}\Big|_{(0,0)} x + \frac{\partial z(x,y)}{\partial y}\Big|_{(0,0)} y - (z - z_{i,j}) = 0 \tag{32}$$

Then the distance from point $M(x, y, z(x, y))$ to this plane will be equal to

$$d(M) = \frac{\left|\partial z(x,y)/\partial x\,|_{(0,0)} x + \partial z(x,y)/\partial y|_{(0,0)}\, y - (z - z_{i,j})\right|}{\sqrt{\left(\partial z(x,y)/\partial x\,\Big|_{(0,0)}\right)^2 + \left(\partial z(x,y)/\partial y\,\Big|_{(0,0)}\right)^2 + 1}}. \tag{33}$$

The points of the surface which are removed from the tangent plane be a distance of no more than ε will be described by inequality

$$d(M) \le \varepsilon \tag{34}$$

Taking into account Taylor's formula for $z = z(x, y)$ in the neighborhood of point (0,0)

$$z(x,y) = z_{i,j} + \frac{\partial z(x,y)}{\partial x}\Big|_{(0,0)} x + \frac{\partial z(x,y)}{\partial y}\Big|_{(0,0)} y + \tag{35}$$

$$\frac{1}{2!}\left(\frac{\partial^2 z(x,y)}{\partial x^2}\Big|_{(0,0)} x^2 + 2\frac{\partial^2 z(x,y)}{\partial x \partial y}\Big|_{(0,0)} xy + \frac{\partial^2 z(x,y)}{\partial^2 y}\Big|_{(0,0)} y^2\right) + O((x+y)^3),$$

we have

$$d(M) = \Big| z_{i,j} + \frac{\partial z(x,y)}{\partial x}\Big|_{(0,0)} x + \frac{\partial z(x,y)}{\partial y}\Big|_{(0,0)} y -$$

$$z_{i,j} - \frac{\partial z(x,y)}{\partial x}\Big|_{(0,0)} x - \frac{\partial z(x,y)}{\partial y}\Big|_{(0,0)} y -$$

$$\frac{1}{2!}\left(\frac{\partial^2 z(x,y)}{\partial x^2}\Big|_{(0,0)} x^2 + 2\frac{\partial^2 z(x,y)}{\partial x \partial y}\Big|_{(0,0)} xy \frac{\partial^2 z(x,y)}{\partial^2 y}\Big|_{(0,0)} y^2\right) + O((x+y)^3)\Big| \times$$

$$\frac{1}{\sqrt{\left(\partial z(x,y)/\partial x\Big|_{(0,0)}\right)^2 + \left(\partial z(x,y)/\partial y\Big|_{(0,0)}\right)^2 + 1}} =$$

$$\frac{1}{2}\left|Ax^2 + 2Bxy + Cy^2\right| \le \varepsilon + O((x+y)^3). \tag{36}$$

Therefore the set of points (x, y) such that values $z(x, y)$ are removed from the tangent plane at point (0,0) by a distance of no more than ε is described by the inequality

$$\frac{1}{2}\left|Ax^2 + 2Bxy + Cy^2\right| \le \varepsilon + O((x+y)^3). \tag{37}$$

It is not difficult to see that this inequality defines the interior of an ellipse with semi-axes $1/\sqrt{2\lambda_1}$ and $1/\sqrt{2\lambda_2}$, where λ_i, the eigenvalues of the quadratic form $Ax^2 + 2Bxy + Cy^2$, are the solutions of the equation

$$\begin{vmatrix} A - \lambda & B \\ B & C - \lambda \end{vmatrix} = 0, \tag{38}$$

and

$$\lambda_i = \frac{C + A \pm \sqrt{4B^2 + (C-A)^2}}{2} \tag{39}$$

One can see from the construction that the given polygonal function is a covering of surface $z = z(x,y)$ by hexagons (Fig. 7). To prove the first part of the theorem, it suffices to show that the top of each hexagon lies at least in some ellipse

$$\begin{aligned} &\frac{1}{2}|A(x_{i,j}, y_{i,j})(x - x_{i,j})^2 + 2B(x_{i,j}, y_{i,j})(x - x_{i,j})(y - y_{i,j}) + \\ &C(x_{i,j}, y_{i,j})(y - y_{i,j})^2| \le \varepsilon + O(((x - x_{i,j}) + (y - y_{i,j}))^3), \end{aligned} \tag{40}$$

where $M_{i,j} = (x_{i,j}, y_{i,j}, z(x_{i,j}, y_{i,j}))$ is a tangent point of the plane. Consequently, we show that this point lies within the $\varepsilon(1 + o(1))$-layer of the surface. For example consider the vertex that is formed by the intersection of tangent planes at points $\Gamma_{i,j}$, $\Gamma_{i+1/2,j+1/2}$ and $\Gamma_{i,j+1}$. This point is defined by the system:

$$\begin{cases} p_{i,j}(x - x_{i,j}) + q_{i,j}(y - y_{i,j}) + z - z_{i,j}) = 0; \\ p_{i+1/2,j+1/2}(x - x_{i+1/2,j+1/2}) + q_{i+1/2,j+1/2}(y - y_{i+1/2,j+1/2}) \\ +z - z_{i+1/2,j+1/2}) = 0; \\ p_{i,j+1}(x - x_{i,j+1}) + q_{i,j+1}(y - y_{i,j+1}) + z - z_{i,j+1}) = 0 \end{cases} \tag{41}$$

where

$$p_{i,j} = \frac{\partial z(x,y)}{\partial x}\Big|_{(x_{i,j}, y_{i,j})}; \tag{42}$$

$$q_{i,j} = \frac{\partial z(x,y)}{\partial y}\Big|_{(x_{i,j}, y_{i,j})}, \tag{43}$$

etc. Solving this system and using a decomposition on Taylor's formula in the neighborhood of point $\Gamma_{i,j}$, we can to verify that at this point

$$\begin{aligned} &\frac{1}{2}\Big|A(x_{i,j}, y_{i,j})(x - x_{i,j})^2 + 2B(x_{i,j}, y_{i,j})(x - x_{i,j})(y - y_{i,j}) + \\ &C(x_{i,j}, y_{i,j})(y - y_{i,j})^2\Big| = \varepsilon(1 + o(1)), \end{aligned} \tag{44}$$

which demonstrates that the polygonal function in the neighborhood of this point lies in the ε-layer of the surface.

The proof that the given method yields asymptotically optimal results is rather difficult. The idea of this proof, without detailed analysis, is based on the following reasoning: first, we approximate the given surface by elliptical paraboloids on the domains for which diameters are on the order of $\varepsilon^{3/2}$. Then each of these paraboloids is approximated by a best polygonal function. In that case, the projections of hexagons of the polygonal function on the plane XOY (which don't touch the boundaries) will be congruent, and the problem will be reduced to the packing of the plane by congruous hexagons; the error of approximation on each of these patches will be ε. Then it may be seen that their number asymptotically equals the number of hexagons yielded by the algorithm under consideration.

It then follows that for surfaces consisting of pieces of elliptical paraboloids, there is no better polygonal function. It remains to be noticed that any sufficiently smooth convex upward surface may be approximated by a surface composed of elliptical paraboloids (using the same number) with an accuracy of $\varepsilon^{3/2}$, and this surface may be used as an intermediate approximation.

Schematically, note how we prove the third part of theorem, i.e. that we cannot construct a polygonal function with a smaller number of planes. Divide the domain D into N parts D_i with approximately equal areas, where

$$N = \left[\frac{1}{\sqrt[3]{\varepsilon^2}} \right] + 1 \tag{45}$$

and on every i-th piece choose a point (x_i, y_i) approximately near the center of gravity of this part and calculate A_i, B_i and C_i at these points and $a_i = 1/\sqrt{2\lambda_1}$ and $b_i = 1/\sqrt{2\lambda_2}$ (where λ_i – are the eigenvalues of quadratic form $A_i x^2 + 2B_i xy + C_i y^2$) or the solutions of characteristic equation (38) (if $A = A_i$, $B = B_i$ and $C = C_i$). Now, for this piece, substitute for z=z(x,y) an elliptic paraboloid

$$z = \frac{(x - \tilde{x}_i)^2}{(a_i)^2} + \frac{(y - \tilde{y}_i)^2}{(b_i)^2} \tag{46}$$

or

$$z = \frac{(x - \tilde{x}_i)^2}{(b_i)^2} + \frac{(y - \tilde{y}_i)^2}{(a_i)^2}, \tag{47}$$

where $\tilde{x}_i$, $\tilde{y}_i$ is some point from D_i. By doing this, the error of description will be

$$O\left(\frac{1}{N^3}\right) = O(\varepsilon^2) \tag{48}$$

and the proposed algorithm will yield an asymptotically optimal covering of this paraboloid by a polygonal function.

It is possible to verify that for every D_i the polygonal function will consist of congruous hexagons in which the most distant points (all vertices of hexagons) are removed from the paraboloid by $\varepsilon(1 + o(1))$. So, in that manner, D_i will be densely packed by these hexagons, so D_i cannot be filled by a smaller number of hexagons. As the description by paraboloids is accurate to $O(\varepsilon^2)$ and by the polygonal function to $O(\varepsilon)$, that proves the third part of the theorem – i.e., that the given algorithm is asymptotically optimal.

We have proved the analog of theorem A for the case of using interpolated piecewise curves. By use of this theorem (analogously to the use of theorem A), we can construct an algorithm for definition of an optimal polygonal function that consists of simplexes, which approximate a convex-up surface from within, and a convex-down (concave) surface from outside. We will discuss this problem in detail in a future article.

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Short Biographical Sketches.

Anatoly A.Ligun
Dr. Ligun is professor and chairperson of higher mathematics at Dnieprodzerzhinsk State Technical University. He was born in 1947, and received the the M.Sc. from Dniepropetrovsk State University (Ukraine) in 1971, and the Ph.D. in 1974. He received the D.Sc. from Kiev Research Institute of Mathematics in 1985. He works is on the theory of function approximation and its applications.

Alexander A.Shumeiko
Dr. Shumeiko is an associate professor of higher mathematics at Dnieprodzerzhinsk State Technical University. He was born in 1955, and received the the M. Sc. in 1977 and the Ph.D. in 1984, both from Dniepropetrovsk State University (Ukraine). His work is in the theory of function approximation and its applications.

Stepan P.Radzevitch
Dr. Radzevitch is professor and dean of the faculty of mechanical engineering of Dnieprodzerzhinsk State Technical University. He was born in 1953, and received the M.Sc. in 1976, and the Ph.D. in 1982, from Kiev Polytechnic Institute. He received the D.Sc. in 1991 from Tula Polytechnic Institute. His primary interest is in mathematical description of the generation of sculptured surface parts.

Erik D.Goodman
Dr. Goodman is professor of electrical engineering and of mechanical engineering, and Director of the Case Center for Computer-Aided Engineering and Manufacturing and of the MSU Manufacturing Research Consortium. Dr. Goodman received the Ph.D. in Computer and Communication Sciences from the University of Michigan in 1972. Dr.Goodman has conducted research in CAD/ CAM/ CAE, environmental modeling, and genetic algorithms for twenty-five years. He has developed numerous software packages dealing with CAD and CAM of sculptured surfaces, for a variety of industrial sponsors. He is co-director of MSU's Genetic Algorithms Research and Applications Group, and its Russian branch, the Russian GARAGe. He is author of the genetic algorithms toolkit "GALOPPS"now being used throughout the world.

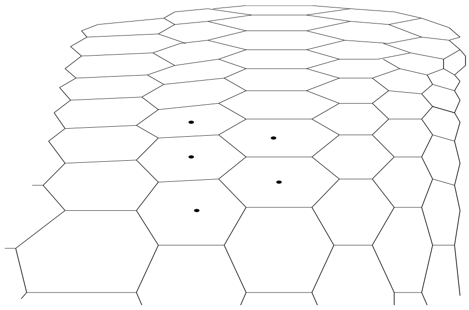


Рис. 1

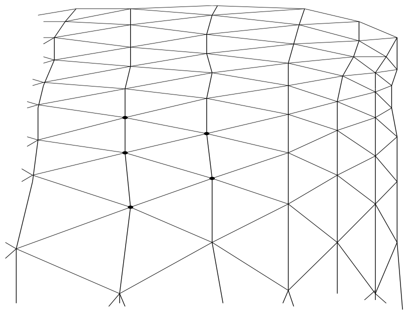
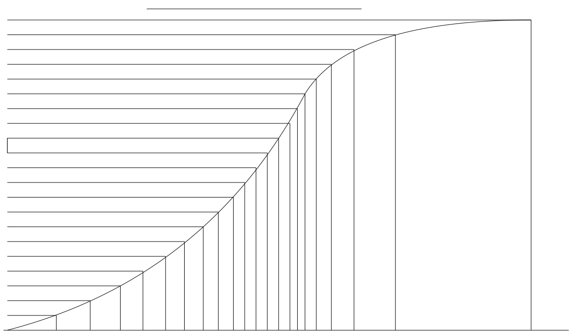


Рис. 2



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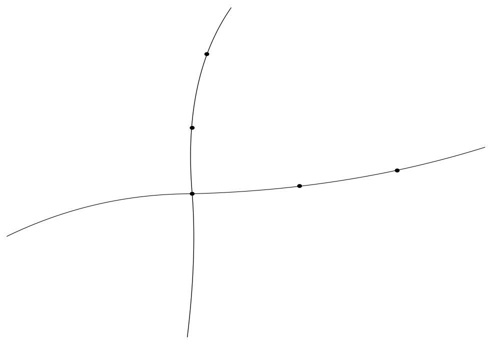

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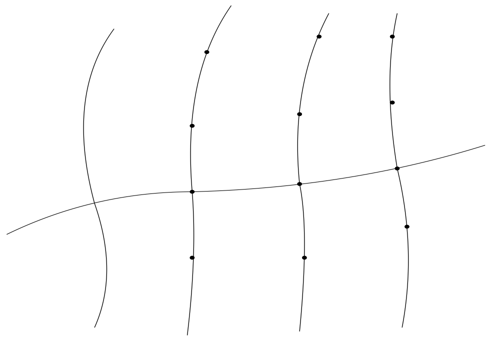

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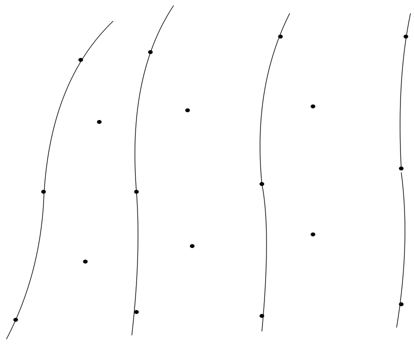

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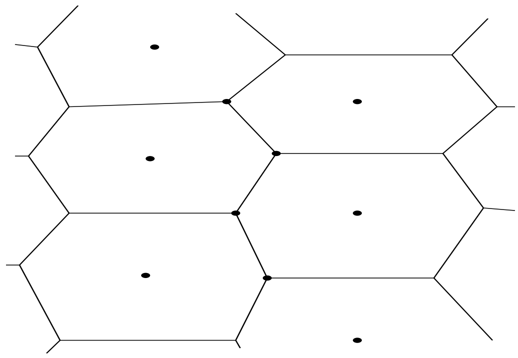

Рис. 7