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ASYMPTOTICALLY OPTIMUM RECOVERY OF SMOOTH CONTOURS BY BÉZIER CURVE

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Abstract In some problem domains, it is important to minimize the visually perceived distance between a given curve and a Bézier curve approximating it. This paper provides an algorithm for constructing a parabolic Bézier curve approximation which deviates the least (asymptotically) from the given curve, according to the Hausdorff metric, which corresponds to minimal deviation as perceived by the human eye.

The rapid development of new manufacturing technologies has ever stronger demands for effective methods of geometric description. In the last decade, alongside with classical methods of curve description - algebraic polynomials and trigonometric polynomials-product and tool engineering have seen many new methods, based primarily on polynomial splines and their numerous generalizations. These methods, although possessing good approximation and computational properties, are not always convenient for solving the tasks of engineering geometry associated with real-world tasks and their technological constraints . Among the most widely known methods for description of smooth contours in geometrical design are the Bézier functions [1] – [4]. As a rule, quadratic Bézier functions are used when there is one intermediate point between initial and final points of interpolation, and cubic, if there are two intermediate points. If there are more points , Bézier functions are applied by reducing the task to one of the two previous cases. The widespread use of Bézier functions results from their ease of construction. On the other hand, the fact that the intermediate points may relatively far from the Bézier curve sometimes limits the use of Bézier functions for the description of contours.

In this paper, an algorithm is given for construction of a parabolic Bézier function which provides the least deviation (asymptotically) from the approximated curve according to the Hausdorff (visual) metric. The use of Hausdorff distance is explained by its convenience for solution of those tasks in which distance is defined visually- i.e., as it is perceived by the human eye [5].

As usual, we define Hausdorff distance between two sets L and Γ by the equations:

$$\rho(L,\rho) = max\left\{\sup_{M\in\Gamma}\inf_{N\in L}|MN|, \sup_{N\in L}\inf_{M\in\Gamma}|MN|\right\},$$

Where $|MN|$ is Euclidean distance between points M and N.

Let $\Gamma(t)$ is smooth flat curve, given in its natural parametrization $\Gamma(t) = (x(t), y(t))$, $t \in [0,T]$, i.e., parameter t is length, beginning at some fixed point, satisfying the conditions:

$$(x'(t))^2 + (y'(t))^2 \equiv 1 \quad t \in [0,T],$$

Let

$$\Delta_n = \{0 = t_1 < t_2 < \ldots < t_{2n} = T\}-$$

Date:

for any partition of the range of the parameter. Then, if

$$h_{i+1/2} = t_{i+1} - t_i \quad (i = 0, 1, \dots, 2n-1),$$

and

$$\Gamma_i = (x_i, y_i) = (x(t_i), y(t_i)) \quad (i = 0, 1, \dots, 2n),$$

for $t \in (t_{2i}, t_{2i+2})$,a parabolic parametrically defined Bézier function will be of the form: $B\Gamma(t) = (Bx(t), By(t))$, where

$$B\Gamma(t) = (1-\tau)^2\Gamma_{2i} + 2\tau(1-\tau)\Gamma_{2i+1} + \tau^2\Gamma_{2i+2},$$

and

$$\tau = \frac{t - t_{2i}}{h_{2i+3/2} + h_{2i+1/2}}.$$

We consider the set of data

$$\tilde{\Gamma}_{2i+1} = \Gamma_{2i+1} - \frac{1}{2}\Delta^2\Gamma_{2i+1} \quad (i = 0, 1, \dots, n-1),$$

where

$$\Delta^2\Gamma_{2i+1} = \Gamma_{2i} - 2\Gamma_{2i+1} + \Gamma_{2i+2}.$$

Then we designate

$$B\tilde{\Gamma}(t) = (Bx(t), By(t)),$$

where

$$B\tilde{\Gamma}(t) = (1-\tau)^2\Gamma_{2i} + 2\tau(1-\tau)\tilde{\Gamma}_{2i+1} + \tau^2\Gamma_{2i+2} = \tag{1}$$

$$(1-\tau)^2\Gamma_{2i} + 2\tau(1-\tau)\left(\Gamma_{2i+1} - \frac{1}{2}\Delta^2\Gamma_{2i+1}\right) + \tau^2\Gamma_{2i+2}.$$

The parabolic Bézier function corresponds to each number i (or to interval $[t_{2i}, t_{2i+2}]$). So number n equals to Bézier functions number, which form the parabolic Bézier spline on interval $[0, T]$.

We will call the partitions Δ_n^0 optimum if

$$\rho(\Gamma, B\tilde{\Gamma}(\Delta_n^0)) = \inf_{\Delta_n} \rho(\Gamma, B\tilde{\Gamma}(\Delta_n)).$$

The sequence partitions Δ_n^* will be called asymptotically optimum if as $n \to \infty$

$$\rho(\Gamma, B\tilde{\Gamma}(\Delta_n^*)) = \inf_{\Delta_n} \rho(\Gamma, B\tilde{\Gamma}(\Delta_n))(1 + o(1)).$$

We define function

$$F(t) = k'(t) = x'(t)y'''(t) - x'''(t)y'(t),$$

where $k(t)$ is the curvature of the curve Γ.

Let $\{F_n(t)\}$ a sequence of functions such that

$$\|F - F_n\|_\infty \le n^{-\alpha},$$

where $\alpha \in (0, 1/4)$

Theorem 1. *Let parametrically defined curve $\Gamma(t) = (x(t), y(t))$, $t \in [0, T]$ be such that x, y be four times continuously differentiable on $[0, T]$ ($x, y \in C^4_{[0,T]}$), number $\alpha \in (0, 1/4)$ and the sequence of partitions $\{\Delta_n^*\}$ is defined by equations:*

$$t^*_{2i} = \theta_i, \quad t^*_{2i+1} = (\theta_{i+1} + \theta_i)/2, \tag{2}$$

where the points θ_i are found according to conditions:

$$\int_0^{\theta_i}(|F_n(t)|^{1/3}+n^{-\alpha})dt=\frac{i}{n}\int_0^T(|F_n(t)|^{1/3}+n^{-\alpha})dt \tag{3}$$

$$(i=0,1,\ldots,n).$$

Then the sequence of partitions $\{\Delta_n^*\}$ will be asymptotically optimum as $n\to\infty$, and

$$\rho(\Gamma,B\tilde{\Gamma}(\Delta_n^*))=\frac{1}{9\sqrt{3}n^3}\left(\int_0^T|F_n(t)|^{1/3}dt\right)^3+o(n^{-3}).$$

In a number of cases, it is the inverse problem which must be solved, i.e., given an existing error ε of the description curve, it is necessary to construct an approximative apparatus with a minimum number of nodes.

Let ε be a given error in the description of a contour. We shall call a Bézier curve $B\tilde{\Gamma}(t)$ ε- allowable for curve $\Gamma(t)$ if

$$\rho(\Gamma,B\tilde{\Gamma})\le\varepsilon.$$

A ε- allowable curve $B\tilde{\Gamma}(\Delta^0_{n_0},t)$ is called ε-иoptimum if, for all ε- allowable curves $B\tilde{\Gamma}(\Delta_n,t)$ the following inequality holds:

$$N_o\le n.$$

The sequence of ε- allowable curves $\{B\tilde{\Gamma}(\Delta^*_{n_*},t)\}$ will be called asymptotically ε optimum if

$$n_*=n_o(1+o(1)).$$

Theorem 2. *Let the parametrically defined curve $\Gamma(t)=(x(t),y(t))$, $t\in[0,T]$ be such that $x,y\in C^4_{[0,T]}$, and let the number n_* be defined by conditions:*

$$n_*=\left[\frac{1}{\sqrt[3]{9\sqrt{3}\varepsilon}}\int_0^T|F^{1/3}(t)|dt\right]+1, \tag{4}$$

(here $[a]$ denotes the integer part of a) and the sequence of partitions $\{\Delta^*_{n_*}\}$ is defined from equations (2)- (3). Then the sequence $\{B\tilde{\Gamma}(\Delta^*_{n_*},t)\}$ is asymptotically ε-optimal.

Remark 1. *In the theorem formulation in equality (3) instead of function $F_n(t)$, in particular, one can take the function $F(t)$. But then the formulation will be more weak. On practice, the approximated curve is given in the kind of some approximation, that is why the using of function $F(t)$ practically is rarely possible.*

Remark 2. *If function $F(t)$ is continuous and its number of zeros on segment $[0,T]$ is finally, moreover, all zeros are simple (i.e. if $F(\tau)=0$, then $F'(\tau)\neq 0$) then addend $n^{-\alpha}$ in formula (3) may be removed and the nodes of asymptotically optimal partitions sequence will be defined from conditions:*

$$\int_0^{\theta_i}(|F_n(t)|^{1/3})dt = \frac{i}{n}\int_0^T(|F_n(t)|^{1/3})dt \qquad (5)$$
$$(i=0,1,\ldots,n).$$

From the approximation theorem point of view, the case when there is the segment on which function $F(t)$ vanishes:

$$F(t)=0 \quad (t\in[a,b],\quad [a,b]\subset[0,T]) \qquad (6)$$

or that is the same, $k(t)=const$ is practically impossible.

In this case there will be the situations, when maximal length of parameter change interval (at using formulae (5)) will not tend to zero. Then there is no reason to say about asymptotic formulae optimality.

But case (6) is sufficiently wide-spread at engineering problems. The curves with constant curvature (as linear curve and circles) sufficiently frequently meet at drawings. That is why in general case the presence of addend $n^{-\alpha}$ in formulae of nodes choice is necessary. The presence of this addend ensures the solvability of equation (3) at any significance of function $F(t)$.

Remark 3. *If curve $F(t)$ is defined in natural parametrization, the theorem gives us concrete algorithm that is easily programmized. To obtain this algorithm it's necessary to find derivatives, calculate the function $F(t)$ and define nodes from formula (3).*

On practice the function Γ is usually defined in discrete form. In this case it's necessary to construct the discrete analogies of all values, being in theorem formulations. Such algorithm is given in paper end.

Remark 4. *Some graphic illustrations are given on Figures.*

(1) *The approximated curve $\Gamma(t)=(\cos(t)+\ 0.2\sin(4t);\ -\sin(t)-\ 0.2\cos(5t)$ $\cos(5t))$ $\ (t\in[0,2\pi])$ is drawn on Figure 1.*
(2) *The Bézier curve $B\tilde{\Gamma}$ at the case, when nodes are chosen on equidistant partition is drawn on Figure 2. The number of intervals $n=15$.*
(3) *The Bézier curve with nodes chosen from condition (3) is drawn on Figure 3. The number $n=15$ is the same.*
(4) *The nodes by Bézier functions, chosen from conditions (3), are drawn on Figure 4 by means of painting over balls.*
(5) *The graphics illustration of nodes choice from conditions (3) is given on Figure 5.*

Insofar as approximation of smooth curves is concerned, it is important to know to what extent they are smooth, and to have a technique for reconstruction of the curve from the given points. For any curve $L(t)=(u(t),v(t))$, where $u(t)$ and $v(t)$ possess one-sided derivatives at each point (this property is shared by $B\tilde{\Gamma}(\Delta_n^*,t)$), we denote $\dot{L}(t+0)=(\dot{u}(t+0),\,\dot{v}(t+0))$ and, similarly $\dot{L}(t-0)$).

We denote as $\varphi(L,t)$ the smallest angle between vectors $\dot{L}(t+0)$) and $\dot{L}(t-0)$). The angle $\varphi(B\tilde{\Gamma}(\Delta_n^*),t)$ characterizes the degree of non-smoothness for curve $B\tilde{\Gamma}(\Delta_n^*)$ at point t.

Theorem 3 *Let parametrically defined curve* $\Gamma(t) = (x(t), y(t))$, $t \in [0, T]$ *be such that* $x, y \in C^4_{[0,T]}$, *and the sequence of partitions* $\{\Delta^*_n\}$ *be defined by equation (3); then*

$$\max_t \varphi(B\tilde{\Gamma}(\Delta^*_n), t) = O\left(\frac{1}{n^3}\right).$$

If curve Γ *such that* $F(t) > 0$ *at all* $t \in [0, T]$ *and the sequence of partitions* $\{\Delta^\star_n\}$ *is defined by equations*

$$t^\star_{2i} = \theta^\star_i, \quad t^\star_{2i+1} = (\theta^\star_{i+1} + \theta^\star_i)/2,$$

where the points $\theta^\star_i$ *are found from conditions*

$$\int_0^{\theta^\star_i} F^{1/3}(t)dt = \frac{i}{n}\int_0^T F^{1/3}(t)dt \tag{7}$$

$$(i = 0, 1, \ldots, n).$$

Then

$$\varphi(B\tilde{\Gamma}(\Delta^\star_n), t^\star_{2i}) = \frac{A_{2i,n}}{n^3} + O\left(\frac{1}{n^4}\right).$$

where

$$A_{2i,n} = I^3\left(\frac{F'_{2i}}{36} + \frac{\Phi_{2i}}{16F_{2i}}\right),$$

and

$$I = \int_0^T F^{1/3}(t)dt,$$

$$\Phi(t) = x^{(4)}(t)y'(t) - y^{(4)}(t)x'(t).$$

Proof of Theorem 1. From the Taylor formula at a point t_{2i+1} we have for $t \in (t_{2i}, t_{2i+2})$

$$x(t) = x_{2i+1} + x'_{2i+1}(t - t_{2i+1}) + \frac{1}{2}x''_{2i+1}(t - t_{2i+1})^2 +$$
$$\frac{1}{6}x'''_{2i+1}(t - t_{2i+1})^3 + O((h_{2i+1/2} + h_{2i+3/2})^4),$$

then

$$\Delta^2 x_{2i+1} = x_{2i} - 2x_{2i+1} + x_{2i+2} = x'_{2i+1}(h_{2i+3/2} - h_{2i+1/2}) +$$
$$\frac{1}{2}x''_{2i+1}(h^2_{2i+1/2} + h^2_{2i+3/2}) + \frac{1}{6}x'''_{2i+1}(h^3_{2i+3/2} - h^3_{2i+1/2}) + O((h_{2i+1/2} + h_{2i+3/2})^4),$$

Substituting the representations just obtained, we have

$$B\tilde{x}(t) = (1-\tau)^2 x_{2i} + 2\tau(1-\tau)\tilde{x}_{2i+1} + \tau^2 x_{2i+2} =$$
$$(1-\tau)^2 x_{2i} + 2\tau(1-\tau)\left(x_{2i+1} - \frac{1}{2}\Delta^2 x_{2i+1}\right) + \tau^2 x_{2i+2} =$$
$$x_{2i+1} + x'_{2i+1}(-(1-\tau)^2 h_{2i+1/2} - \tau(1-\tau)(h_{2i+3/2} - h_{2i+1/2}) + \tau^2 h_{2i+3/2}) +$$
$$\frac{1}{2}x''_{2i+1}((1-\tau)^2 h^2_{2i+1/2} - \tau(1-\tau)(h^2_{2i+3/2} + h^2_{2i+1/2}) + \tau^2 h^2_{2i+3/2}) +$$
$$\frac{1}{3!}x'''_{2i+1}(-(1-\tau)^2 h^3_{2i+1/2} - \tau(1-\tau)(h^3_{2i+3/2} - h^3_{2i+1/2}) + \tau^2 h^3_{2i+3/2}) +$$
$$O((h_{2i+3/2} + h_{2i+1/2})^4).$$

It is clear from this that for $t \in (t_{2i}, t_{2i+2})$

$$x(t) - B\tilde{x}(t) = \frac{3}{2} x'_{2i+1} \tau(\tau - 1)(h_{2i+3/2} - h_{2i+1/2}) -$$
$$\frac{1}{2} x''_{2i+1} \tau(1-\tau)(h_{2i+3/2} - h_{2i+1/2})^2 + \frac{1}{6} x'''_{2i+1} \tau(1-\tau)(\tau(h_{2i+3/2} + h_{2i+1/2})^3 -$$
$$(3h_{2i+3/2} h^2_{2i+1/2} - h^3_{2i+3/2})) + O((h_{2i+3/2} + h_{2i+1/2})^4).$$

The first terms on the right-hand side of the last equation are zero (we shall hereinafter consider only such partitions) whenever

$$h_{2i+1/2} = h_{2i+3/2} = h_i,$$

Under this condition, the error will be of the form

$$x(t) - B\tilde{x}(t) = \frac{4}{3} x'''_{2i+1} \tau(1-\tau)(\tau - 1/2) h_i^3 + O(h_i^4).$$

As

$$x(t) - x(\zeta) = x'(t)\xi + \frac{x''(t)}{2}\xi^2 + \frac{x'''(t)}{6}\xi^3 + O(\xi^4),$$

where $\xi = t - \zeta$, then uniformly on i

$$x(\zeta) - B\tilde{x}(t) = x(t) - B\tilde{x}(t) + x(\zeta) - x(t) =$$
$$\frac{4}{3} x'''_{2i+1} \tau(1-\tau)(\tau - 1/2) h_i^3 - x'(t)\xi - \frac{x''(t)}{2}\xi^2 - \frac{x'''(t)}{6}\xi^3 + O(\xi^4 + h_i^4).$$

In addition, for $t \in (t_{2i}, t_{2i+2})$

$$\rho(\Gamma, \widetilde{B(\Gamma)}) \le$$
$$\sqrt{\|x(t) - B\tilde{x}(t)\|^2_{C[t_{2i}, t_{2i+2}]} + \|y(t) - B\tilde{y}(t)\|^2_{C[t_{2i}, t_{2i+2}]}} = O(h_i^3)$$

holds uniformly on i. Therefore, according to the definition of the Hausdorff distance, we need only consider those ξ which satisfy the condition $\xi = O(h_i^3)$ uniformly on i. Then we obtain

$$x(\zeta) - B\tilde{x}(t) = \frac{4}{3} x'''_{2i+1} \tau(1-\tau)(\tau - 1/2) h_i^3 - x'(t)\xi + O(h_i^4),$$

and also

$$y(\zeta) - B\tilde{y}(t) = \frac{4}{3} y'''_{2i+1} \tau(1-\tau)(\tau - 1/2) h_i^3 - y'(t)\xi + O(h_i^4).$$

Let

$$|\Gamma(\zeta) - B\tilde{\Gamma}(\Delta_n, t)|^2 = (x(\zeta) - B\tilde{x}(t))^2 + (y(\zeta) - B\tilde{y}(t))^2.$$

Then, if $\max h_i \to 0$ as $n \to \infty$, then for n large enough, the equation

$$\max_{\zeta} \min_{t} |\Gamma(\zeta) - B\tilde{\Gamma}(\Delta_n, t)|^2 =$$
$$\max_{i} \max_{0<\tau<1} \left(\left(\frac{4}{3} F_{2i+1} \tau(1-\tau)(\tau - 1/2) h_i^3 \right)^2 + O(h_i^7) \right) =$$
$$\max_{i} \left(\left(\frac{1}{9\sqrt{3}} F_{2i+1} h_i^3 \right)^2 + O(h_i^7) \right).$$

will hold. Similarly,

$$\max_{t} \min_{\zeta} |\Gamma(\zeta) - B\tilde{\Gamma}(\Delta_n, t)|^2 = \max_{i} \left(\left(\frac{1}{9\sqrt{3}} F_{2i+1} h_i^3 \right)^2 + O(h_i^7) \right).$$

Thus, if $\max h_i \to 0$ as $n \to \infty$, then for large enough n, the equation

$$\rho^2(\Gamma, B\tilde{(}\Gamma, \Delta_n)) = \max_i \left(\left(\frac{1}{9\sqrt{3}} F_{2i+1} h_i^3 \right)^2 + O(h_i^7) \right).$$

will hold uniformly on i. Using the last equation, and following the outline of the proof in references [6] – [8], we obtain the statement of the theorem.

The proof of Theorem 2 easily follows from Theorem 1.

Proof of Theorem 3. From the equations derived earlier to be convinced that

$$\sin\varphi(B\tilde{\Gamma}(\Delta_n), t_{2i}) =$$
$$(\frac{1}{3}(x_{2i}'''y_{2i}' - x_{2i}'y_{2i}''')(h_i^2 - h_{i-1}^2) +$$
$$\frac{1}{4}(x_{2i}^{(4)}y_{2i}' - x_{2i}'y_{2i}^{(4)})(h_{i-1}^3 + h_i^3) + O((h_{i-1} + h_i)^4)).$$

Examining choice of nodes, as indicated in a condition of Theorem 3, we receive

$$h_{i-1}^\star = \frac{I}{2n(F_{2i})^{1/3}} + \frac{F_{2i}' I^2}{24n^2 (F_{2i})^{2/3}} + O(n^{-3}),$$

and

$$h_i^\star = \frac{I}{2n(F_{2i})^{1/3}} - \frac{F_{2i}' I^2}{24n^2 (F_{2i})^{2/3}} + O(n^{-3}).$$

From these and previous equations we obtain, for all i,

$$\sin\varphi(B\tilde{\Gamma}(\Delta_n^\star), t_{2i}) = \frac{I^3}{n^3} \left(\frac{F_{2i}'}{36} + \frac{\Phi_{2i}}{16F_{2i}} \right),$$

This completes the proof of the Theorem 3.

We can apply the algorithm, and receive the desired result. Assume that curve is given by a set of points describing the given contour with sufficient precision. It could be a set of data taken manually from a surface, by scanner output, or in some other way (by smoothing them,for example), or a set obtained by updating given points by means of an approximating curve (for example, a spline).

So, let N points of smooth curve be given:

$$M_i = (x_i, y_i) \quad (i = 1, 2, \ldots, N),$$

and let the allowable error be ε.

We calculate quantities:

$$l_{i+1/2} = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} \quad (i = 1, 2, \ldots, N-1),$$

and

$$\sigma_i = \sum_{\nu=1}^{i} l_{\nu+1/2}, \ (i = 1, 2, \ldots, N-1).$$

We calculate

$$F_{i+1/2} = \Delta x_{i+1/2} \Delta^3 y_{i+1/2} - \Delta^3 x_{i+1/2} \Delta y_{i+1/2} \ (i = 1, \ldots, N-1),$$

and

$$F_{1/2} = F_{3/2}; \quad F_{N-1/2} = F_{N-3/2}.$$

We now calculate a discrete analog of the primitive:

$$\Psi(t) = \int_0^t F(t)^{1/3} dt$$

It will be

$$\Psi_i = \sum_{\nu=1}^{i} (|F_{\nu+1/2}|^{1/3} + N^{-1}) l_{\nu+1/2},\ (i = 1, 2, \ldots, N-1),$$

$$\Psi_0 = 0.$$

Further, using the formula (4) from theorem 2 we calculate the asymptotically optimum number of nodes,according to the algorithm, as

$$n = \left[\frac{6}{\sqrt[3]{9\sqrt{3}\varepsilon}} \Psi_{N-1} \right] + 1,$$

And we construct the nodes of the asymptotically optimum partition t_i $(i = 1, 2, \ldots, 2n+1)$ according to the algorithm indicated in Theorem 1:

$$t_j = (j-1)\Psi_{N-1}/(2n+1) - \Psi_{k-1}/\Psi_k + (k-2)\sigma_{N-1}/N$$

where

$$\frac{i\sigma_{N-1}}{N} \le t_j \le \frac{(i+1)\sigma_{N-1}}{N}$$

and

$$\Psi_k < (j-1)\Psi_N/(2n+1)$$

for $k = 2, 3, \ldots$ and $i \le 2n+1$.

The task remains to calculate discrete analogs for

$$(x(t_j), y(t_j))\ (j = 1, \ldots, 2n+1)$$

For $t_j \in (\sigma_i, \sigma_{i+1})$, there are data:

$$x(t_j) = \left(\frac{\sigma_{i+1} - t_j}{l_{i+1/2}} \right) x_i + \left(\frac{t_j - \sigma_i}{l_{i+1/2}} \right) x_{i+1},$$

$$y(t_j) = \left(\frac{\sigma_{i+1} - t_j}{l_{i+1/2}} \right) y_i + \left(\frac{t_j - \sigma_i}{l_{i+1/2}} \right) y_{i+1}.$$

We construct the Bézier function with the required properties using the information obtained by applying formulae (1).

Список литературы

[1] *Boehm, W., Farin, G.E. and Kahman,J.(1989).* A survey of curve and surface methods, Computer-Aided Geometric Design, 1-60.

[2] *Farin, G.E.* Curves and Surfaces for Computer-Aided Geometric Design. A Practical Guide. Academic Press, 1990, 444pp.

[3] *Barry, P. and Goldman, R.* De Casteljau-type subdivision in peculiar to Bézier curves, Computer-Aided Design 20(3): 144-116,1988.

[4] *Boehm, W.* Generating the Bézier points of B-Splines. Computer-Aided Design 13(6), 1981.

[5] *Sendov Bl.(1990)* Hausdorff Approximations. Kluwer Academic Publishers, Dordrecht /Boston/ London.

[6] *Ligun A.A., Shumeiko A.A.* Optimum choice of nodes at approximation of functions by splines.- Dokl. AN UkrSSR, ser.A, 1984, N6.

[7] *Ligun A.A., Shumeiko A.A.* On optimum choice of nodes at approximation of functions by interpolation splines.- J.of calc. math. and math. ph., 1984, Ў.24, N9.

[8] *Ligun A.A., Shumeiko A.A.* On one property of splines with asymptotically optimum nodes.- In coll. "Questions of optimum approximation of functions and summation of serieses Dnepropetrovsk State University, 1988, б.41-47.

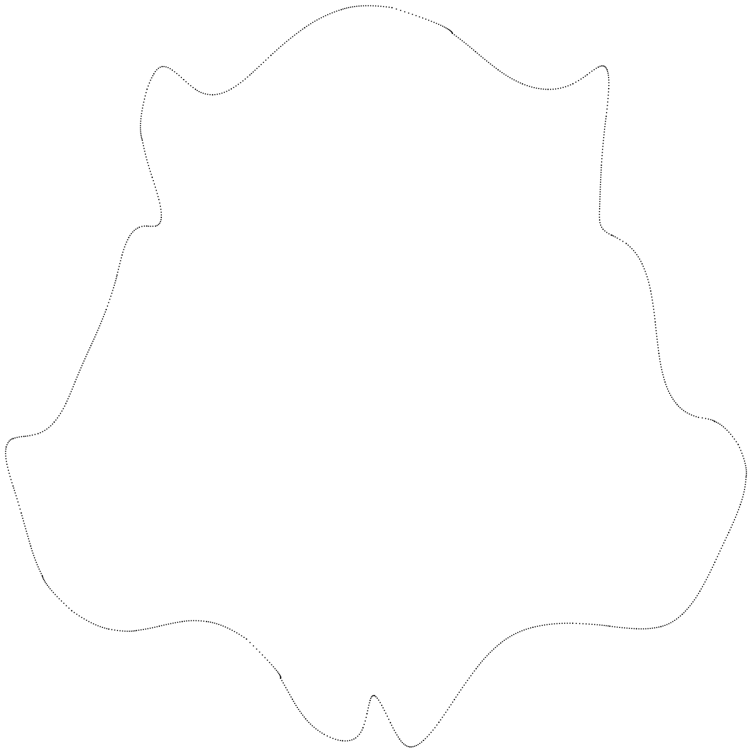

Рис. 1. The approximated curve

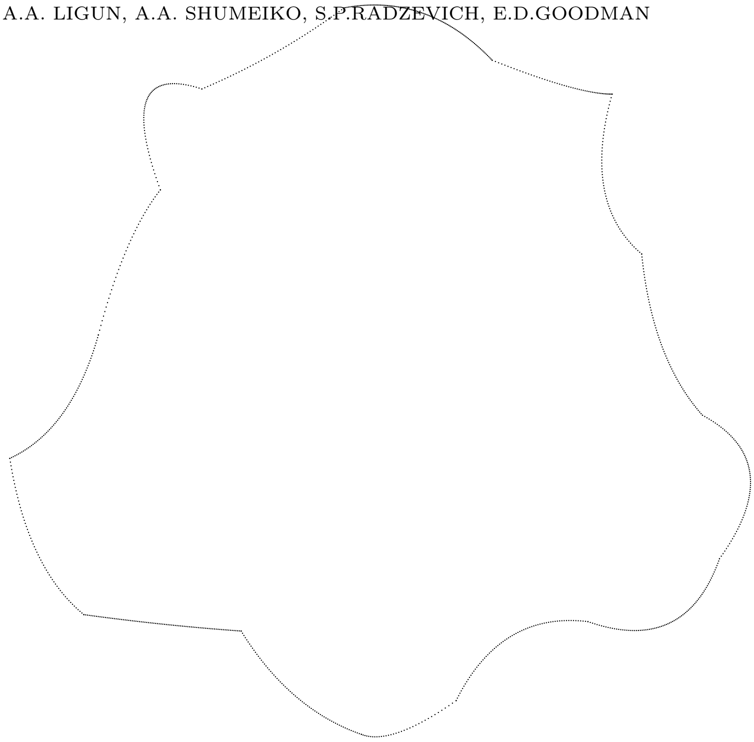

Рис. 2. Bézier curve with equidistant nodes. The number of intervals $n = 15$

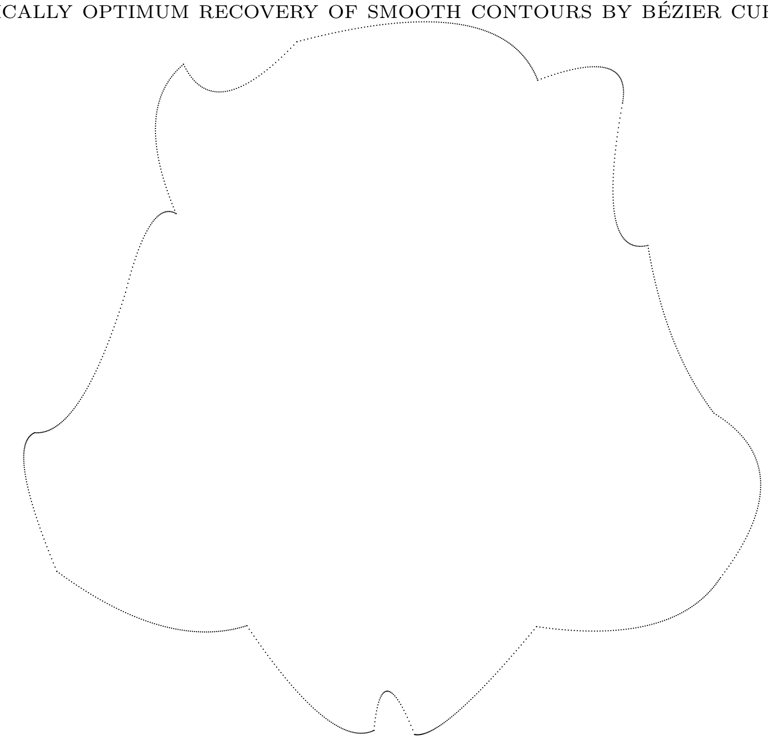

Рис. 3. The Bézier curve with asymptotically optimal nodes. The number of intervals $n = 15$.

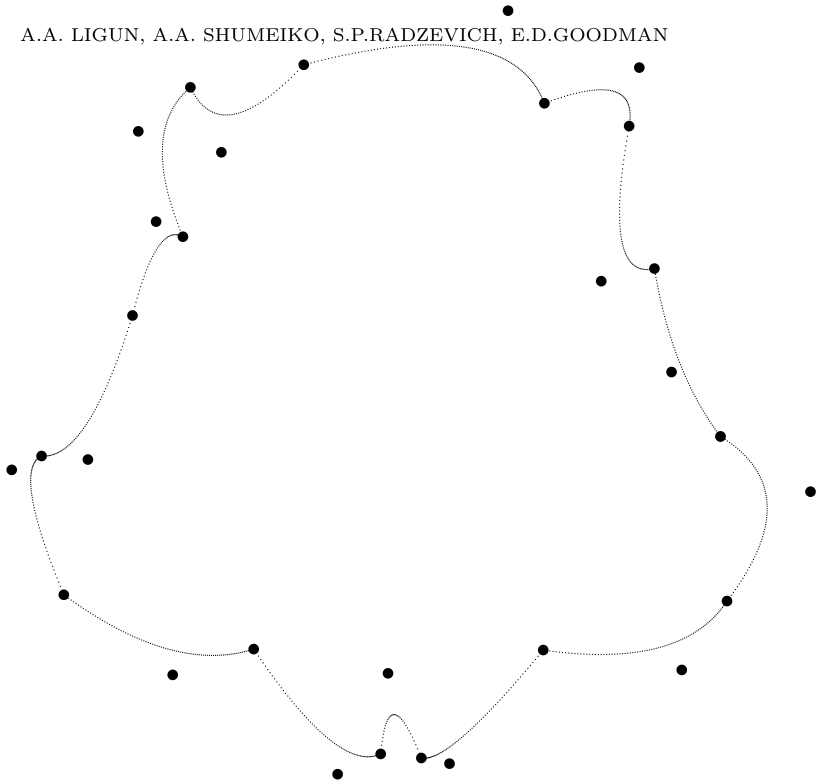

Рис. 4. The symptotically optimal nodes by Bézier curve if number $n = 15$.

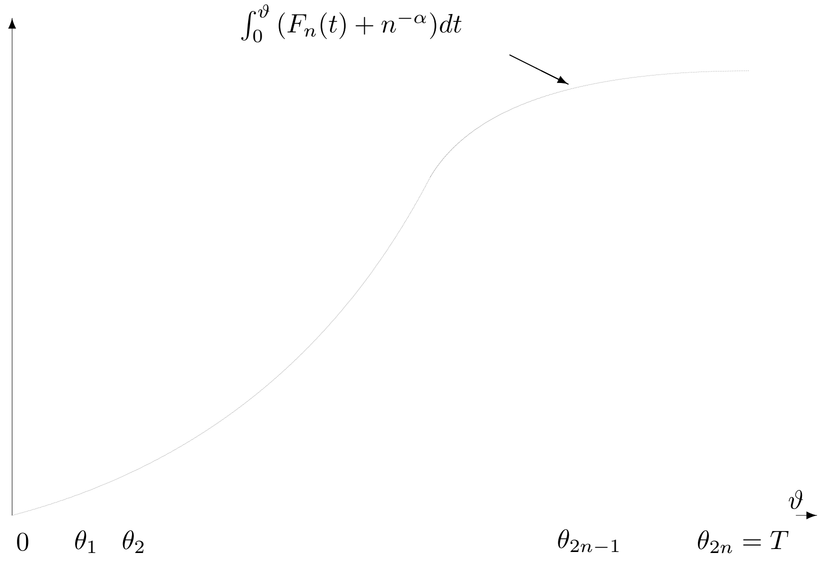
$\int_0^\vartheta (F_n(t) + n^{-\alpha})dt$
ϑ
0
θ_1
θ_2
θ_{2n-1}
$\theta_{2n} = T$

Рис. 5

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