Surface Approximation Using Average Interpolating Splines

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***Abstract*—This work is devoted to creation of local average interpolating splines, which can be further used for solution of various problems, e.g for building surfaces by data from aerial and satellite imagery. An iterative process for obtaining of local splines that converge to average interpolating splines with any given accuracy is created. Examples of application of the obtained splines by example of computer graphics problems are given.**

***Keywords*—splines; tensor product of splines; image resampling; surface approximation**

I. Introduction

One of important areas of UAVs application is the observation of Earth's surface during flight and digital photographing of chosen areas. Automatic flight of UAV depends on information about terrain which is usually presented in vector form. Splines are important tools for complex surfaces approximation with interpolating splines method being one of the most popular methods.

In [1] we have created a method of local almost interpolating splines of two variables. If there is enough initial data, application of interpolating spline built on sparse grid leads to the loss of initial data. To avoid such losses, it is recommended to use math tools which aggregate data. Such tools include splines that preserve the average value of the approximated function.

This work is devoted to the solving of this problem.

Another important problem is analysis of objects discovered during aerial photography which often requires image resizing. This work is concerned with this task too.

II. Surface Approximation

Approximated surface is given on rectangular grid (ih,jH) where $(i,j) \in Z^2$. Biquadratic spline s of minimal defect with knots at the points (ih,jH) is called average interpolating spline for function f with bounded values $f(u,v)$ over cells $(u,v) \in [ih,jh] \times [(i+1)h,(j+1)h]$ if

$$\frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} f(u,v)\,dv\,du = \frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} s(u,v)\,dv\,du.$$

Let us consider surface approximation by tensor product of B-splines defined on a rectangular grid (ih,jH) where $(i,j) \in Z^2$. Surface $S_{2,2}(u,v)$ defined by tensor product of second-order B-splines has the form

$$S_{2,2}(P,u,v) = \sum_{(i,j)\in Z^2} P_{i,j}\cdot N_{2,2}\left(u-\left(i+\frac{1}{2}\right)h, v-\left(j+\frac{1}{2}\right)H\right),$$

where $(u,v) \in R^2$ and $N_{2,2}(u,v) \equiv B_2(u)\cdot B_2(v)$ is a normalized tensor product of second-order B-splines on grid (i,j) which has the following plot (fig. 1).

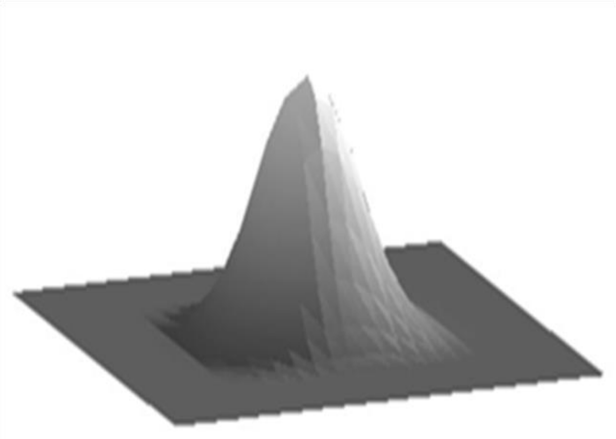

Fig. 1. Plot of function $N_{2,2}(u,v)$.

Accordingly, we obtain the spline formula in matrix form:

$$S_{2,2}(P,u,v) = \begin{bmatrix} 1 & \dfrac{v-ih}{h} & \left(\dfrac{v-ih}{h}\right)^2 \end{bmatrix} M_{2,2} P_{2,2} M_{2,2}^T \begin{bmatrix} 1 \\ \dfrac{u-jh}{h} \\ \left(\dfrac{u-jh}{h}\right)^2 \end{bmatrix} \qquad (1)$$

where

$$M_{2,2} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix},\quad P_{2,2} = \begin{bmatrix} P_{i-1,j-1} & P_{i,j-1} & P_{i+1,j-1} \\ P_{i-1,j} & P_{i,j} & P_{i+1,j} \\ P_{i-1,j+1} & P_{i,j+1} & P_{i+1,j+1} \end{bmatrix}.$$

III. Main Results

As noted before, in each cell of the partition spline must satisfy the following condition:

$$\frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} S_{2,2}(P,u,v)\,dv\,du = \frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} f(u,v)\,dv\,du.$$

Using (1), we obtain that

$$\frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} S_{2,2}\left(P,u,v\right)dv\;du = P_{i,j} +$$
$$+\frac{1}{9}\left(P_{i-1,j}+P_{i,j-1}+P_{i,j+1}+P_{i+1,j}-4P_{i,j}\right)+$$
$$+\frac{1}{36}\left(P_{i-1,j-1}+P_{i-1,j+1}+P_{i+1,j-1}+P_{i+1,j+1}-4P_{i,j}\right)= \tag{2}$$
$$=P_{i,j}+\frac{1}{36}\Delta P_{i,j},$$

where

$$\Delta P_{i,j}=4\left(P_{i-1,j}+P_{i,j-1}+P_{i,j+1}+P_{i+1,j}-4P_{i,j}\right)+$$
$$+\left(P_{i-1,j-1}+P_{i-1,j+1}+P_{i+1,j-1}+P_{i+1,j+1}-4P_{i,j}\right).$$

Let us denote

$$P^*_{i,j}=P^*_{i,j}\left(f\right)=\frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} f\left(u,v\right)dv\;du.$$

Let

$$P_{i,j}=P_{i,j}\left(f\right)=P^*_{i,j}+\sum_{\nu=1}^{\infty}\left(-\frac{1}{36}\right)^{\nu}\Delta^{\nu}P^*_{i,j}, \tag{3}$$

where $\Delta^{2\nu}P^*_{i,j}=\Delta^{\nu}\left(\Delta^{\nu-1}P^*_{i,j}\right)$.

From this and from (2) it follows that

$$\frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} S_{2,2}\left(P,u,v\right)dv\;du=$$
$$=P^*_{i,j}+\sum_{\nu=1}^{\infty}\left(-\frac{1}{36}\right)^{\nu}\Delta^{\nu}P^*_{i,j}+\frac{1}{36}\Delta\left(P^*_{i,j}+\sum_{\nu=1}^{\infty}\left(-\frac{1}{36}\right)^{\nu}\Delta^{\nu}P^*_{i,j}\right)=$$
$$=P^*_{i,j}+\sum_{\nu=1}^{\infty}\left(-\frac{1}{36}\right)^{\nu}\Delta^{\nu}P^*_{i,j}+\frac{1}{36}\Delta^{\nu}P^*_{i,j}+$$
$$+\frac{1}{36}\Delta\left(\sum_{\nu=1}^{\infty}\left(-\frac{1}{36}\right)^{\nu}\Delta^{\nu}P^*_{i,j}\right)=$$
$$=P^*_{i,j}+\sum_{\nu=2}^{\infty}\left(-\frac{1}{36}\right)^{\nu}\Delta^{\nu}P^*_{i,j}-\sum_{\nu=1}^{\infty}\left(-\frac{1}{36}\right)^{\nu+1}\Delta^{\nu+1}P^*_{i,j}=$$
$$=P^*_{i,j}+\sum_{\nu=2}^{\infty}\left(-\frac{1}{36}\right)^{\nu}\Delta^{\nu}P^*_{i,j}-\sum_{\nu=2}^{\infty}\left(-\frac{1}{36}\right)^{\nu}\Delta^{\nu}P^*_{i,j}=$$
$$=P^*_{i,j}=\frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} f\left(u,v\right)dv\;du.$$

Therefore, if spline coefficients $P_{i,j}$ are defined by (3), then for all (i,j)

$$\frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} S\left(P,u,v\right)dv\;du=\frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} f\left(u,v\right)dv\;du.$$

Thus, when spline coefficients are defined by (3) spline $S_{2,2}(P(f),u,v)$ interpolates on average function f over each cell $(u,v)\in[ih,jh]\times[(i+1)h,(j+1)h]$.

Let

$$P^1_{i,j}=P^1_{i,j}\left(f\right)=P^*_{i,j}-\frac{1}{36}\Delta P^*_{i,j},$$

we denote the corresponding spline as

$$S^1_{2,2}\left(P^1,u,v\right)=\sum_{(i,j)\in Z^2}P^1_{i,j}\cdot N_{2,2}\left(u-\left(i+\frac{1}{2}\right)h,v-\left(j+\frac{1}{2}\right)h\right). \tag{4}$$

Then

$$\frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} S^1_{2,2}\left(P^1,u,v\right)dv\;du=P^*_{i,j}-\frac{1}{1296}\Delta^2P^*_{i,j}.$$

If function $f(u,v)$ has all continuous derivatives up to fourth order inclusive, then

$$\frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} S^1_{2,2}\left(P^1,u,v\right)dv\;du=$$
$$=\frac{1}{h\cdot H}\int\limits_{u=ih}^{(i+1)h}\int\limits_{v=jH}^{(j+1)H} f\left(u,v\right)dv\;du-$$
$$-\frac{1}{36}\left(h^4\left.\frac{\partial^4 f}{\partial x^4}\right|_{\left(i+\frac{1}{2},j+\frac{1}{2}\right)}+2h^2H^2\left.\frac{\partial^4 f}{\partial x^2\partial y^2}\right|_{\left(i+\frac{1}{2},j+\frac{1}{2}\right)}+\right.$$
$$\left.+H^4\left.\frac{\partial^4 f}{\partial y^4}\right|_{\left(i+\frac{1}{2},j+\frac{1}{2}\right)}\right)+O\left(\left(max\left\{h,H\right\}\right)^5\right).$$

Thus, spline (4) is an average interpolating spline with precision of $O((max\{h,H\})^4)$ [2].

IV. Example

The proposed method is based on the requirement of the existence of the fourth derivative on the surface. However, it allows one to get satisfactory results even for non-smooth surfaces. As an example, application of the proposed method for image resizing is given.

We consider pixels as cells and color value (for each color component) as the value of f on cell. We build spline by this data and then (using a new grid that is different from original one) get values for new pixels. By repeating this procedure once again we obtain image with original width and height that has been distorted two times by application of splines. Then we can compare the original image with the image which was distorted by splines application.

PSNR (peak signal-to-noise ration) is used as a criterion of images similarity. For two images I, J of one and the same size $H\times W$

$$PSNR(I,J) = 20\log_{10}\frac{255}{\sqrt{\frac{1}{H\times W}\sum_{i=0}^{H-1}\sum_{j=0}^{W-1}\left(I(i,j)-J(i,j)\right)^2}}.$$

Experiments were conducted using image collection TID2008 [3] with image Lenna added. In this article PSNR is given for images that were first doubled in size and then were resized back to original size. One can notice that the average interpolating spline demonstrated better results than the interpolating spline and they both showed better results than the standard B-spline (fig. 2).

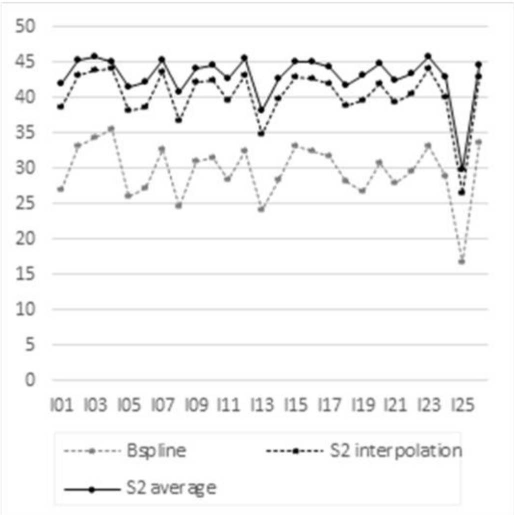

Fig. 2. PSNR comparison for S_2 interpolation, S_2 average and standard B-spline.

Comparison with other resizing methods (fig. 3) demonstrated that only Lanczos resampling produces results that are not inferior to the results obtained by application of our method.

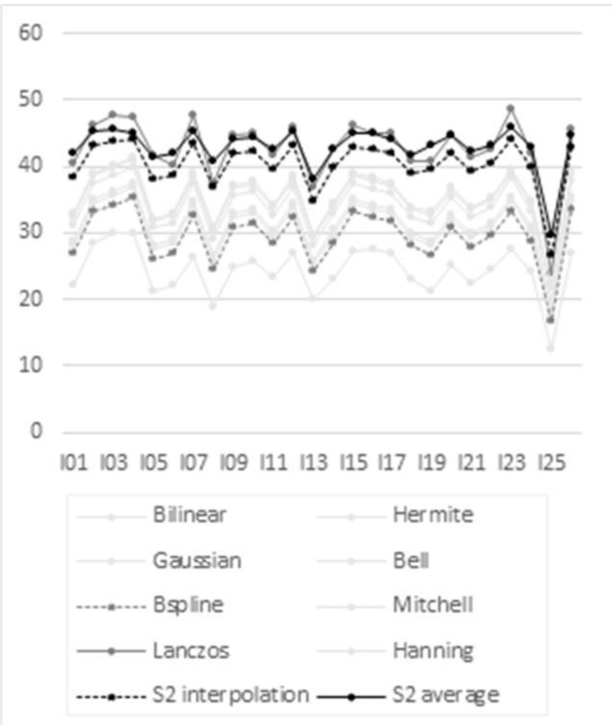

Fig. 3. Comparison of different image resizing methods.

However, Lanczos kernel $\dfrac{a\cdot\sin(\pi\cdot x)\cdot\sin\left(\dfrac{\pi\cdot x}{a}\right)}{\pi^2\cdot x^2}$ has pronouncedly “oscillating” derivatives (fig. 4). Therefore, it is poorly applicable for surface approximation when not only approximation of surface but also approximation of derivatives is needed.

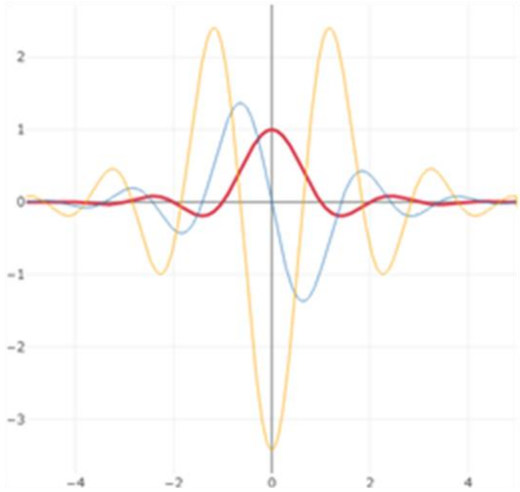

Fig. 4. Lanczos kernel (a=5), its first and second derivatives.

V. Conclusion

The proposed method enables one to create local average almost interpolating biquadratic splines. Experiments have demonstrated good results of application of such splines even for non-smooth surfaces, i.e. in image resizing.

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